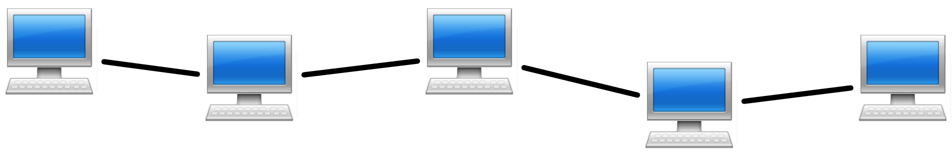
- **Weeks 1–2: informal introduction**
    - network = path



- **Week 3: graph theory**
- **Weeks 4–7: models of computing**
    - what can be computed (efficiently)?
- **Weeks 8–11: lower bounds**
    - what cannot be computed (efficiently)?
- **Week 12: recap**

Week 2

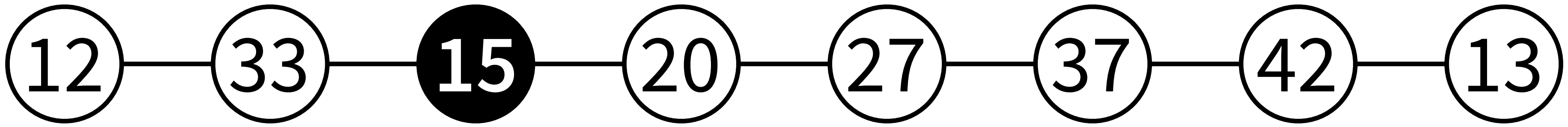
– Warm-up: negative results

Locality

- **Output of a node can only depend on what it knows**
- **After *T* time steps, a node can only know about things up to distance *T***

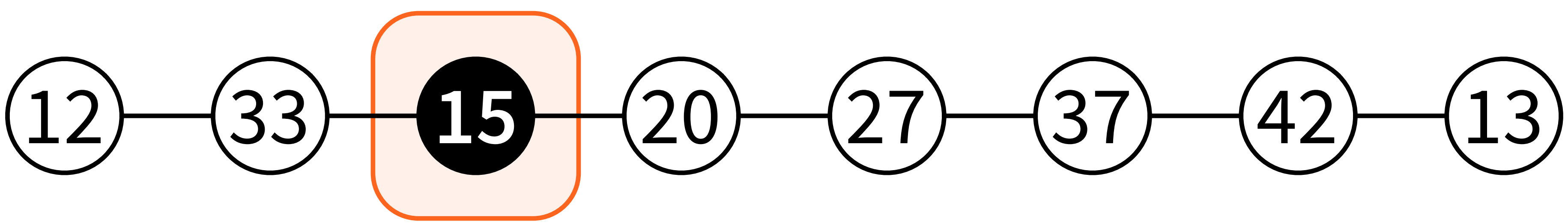
Locality

- **Who knows that node 15 exists?**
  - initially, only node 15
  - everyone else has to learn it
    by exchanging messages



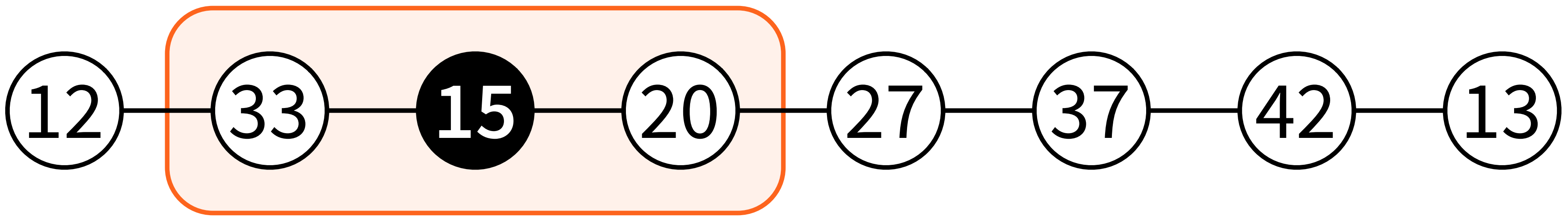
Locality

- **Who knows about node 15 at time $T = 0$?**
  - initial state, before we exchange any messages



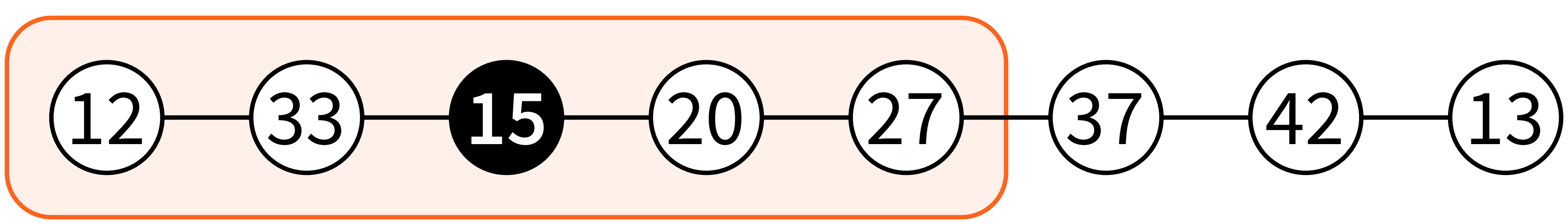
Locality

- **Who knows about node 15 at time *T* = 1?**
  - after 1 communication round



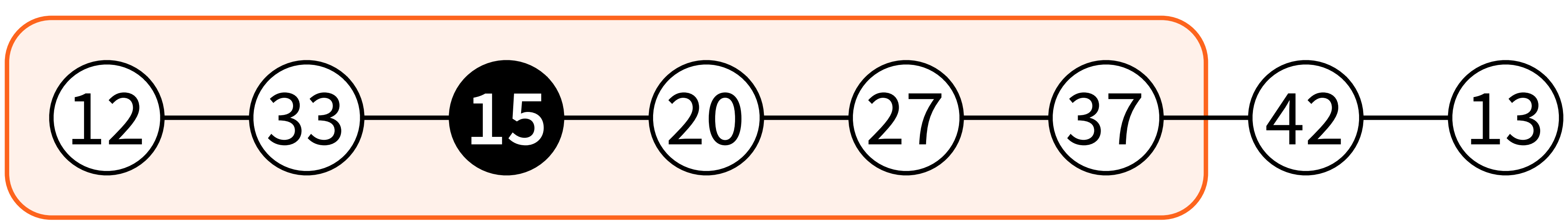
Locality

- **Who knows about node 15 at time $T = 2$?**
  - after 2 communication rounds



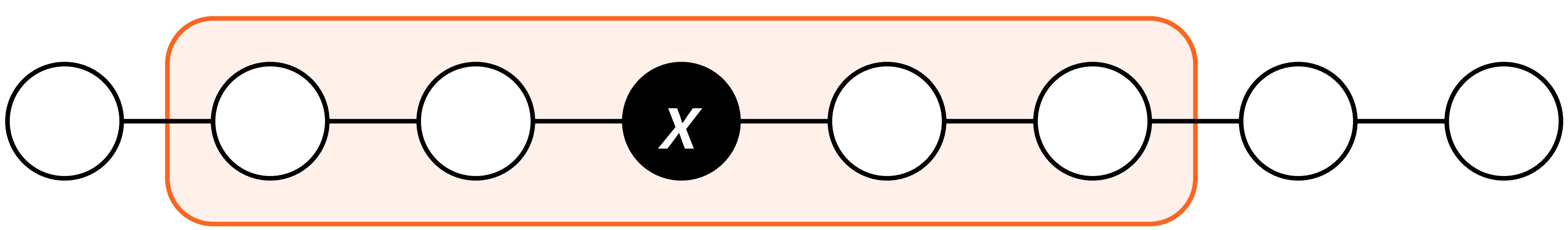
Locality

- **Who knows about node 15 at time *T* = 3?**
  - after 3 communication rounds



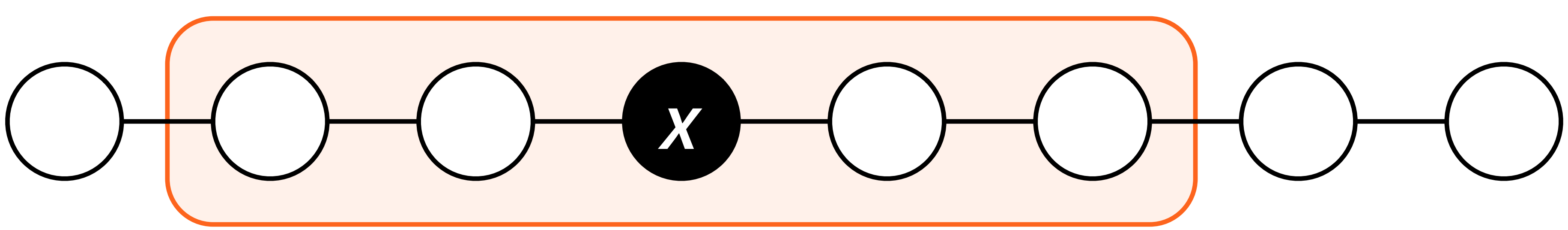
Locality

- **After *T* communication rounds, only nodes up to distance *T* from node *x* can know anything about node *x***
  - distance = “number of hops”



Locality

- **After *T* communication rounds, node *x* can only know about other nodes that are within distance *T* from it**
  - distance = “number of hops”

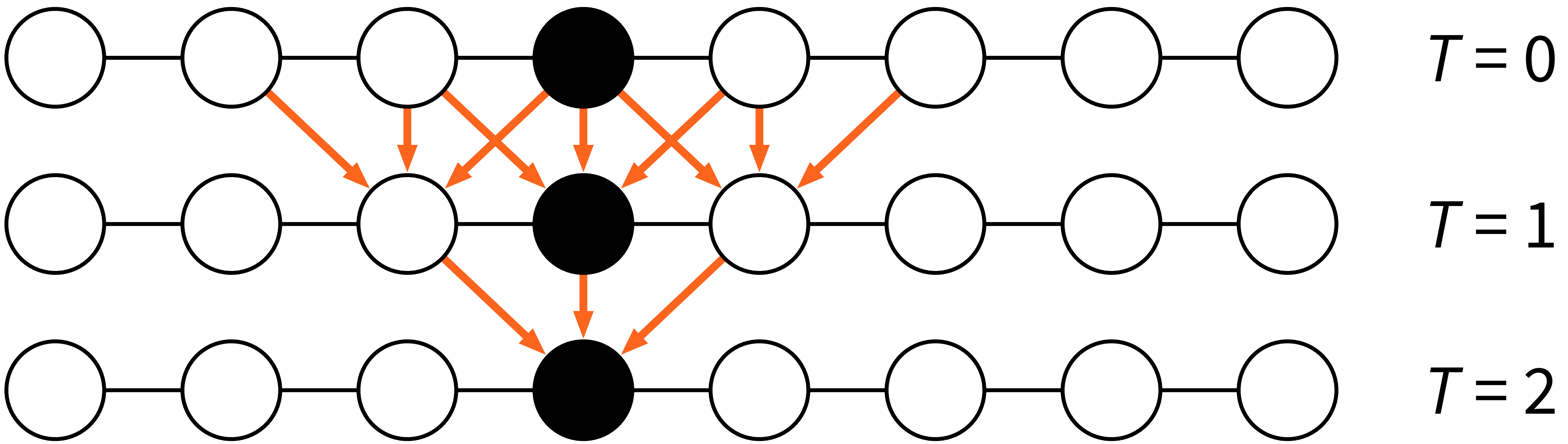


Locality

- **My state at time *T* only depends on:**
  - my state at time $T - 1$, and
  - messages that I received on round T, which only depend on:
    - the state of my neighbours at time $T - 1$

Locality

- **State at time *T* only depends on initial information within distance *T***

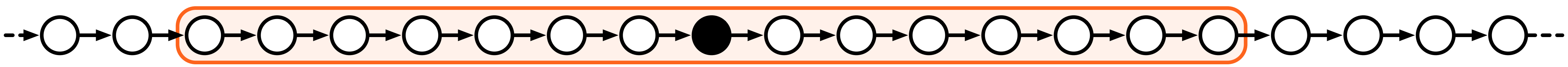


Locality

- **Time = distance**
- **Fast algorithm = “local” algorithm**
    - outputs only depend on local neighbourhoods

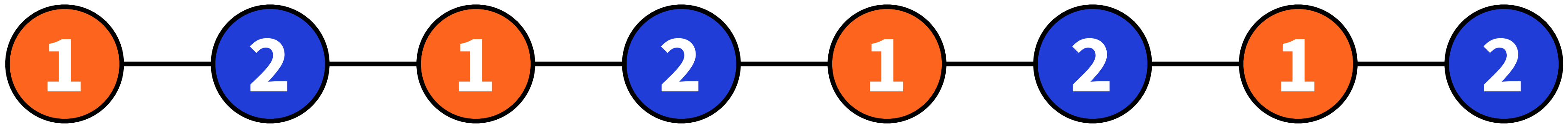
Example: 3-colouring

- Recall: given 128-bit unique identifiers, 3-colouring possible in 7 rounds
- Equivalently: each node can pick its colour based on what it sees in its radius-7 neighbourhood



Using locality to prove lower bounds

- **Example: 2-colouring of a path**
- **Upper bound: possible in time $O(n)$**
- **Lower bound: not possible in time $o(n)$**

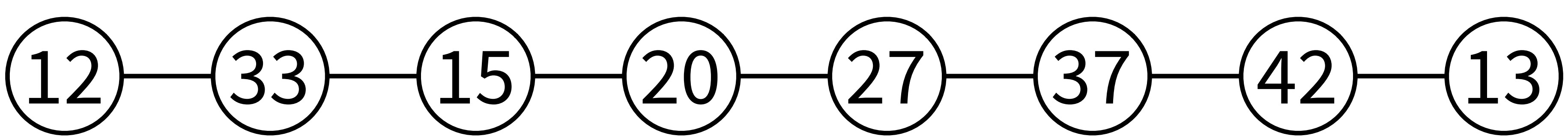


Algorithm for 2-colouring

- **Assumption: path, unique identifiers**
- **Two phases:**
  - find the endpoint with smaller identifier
  - starting from this end, assign colours $1, 2, 1, 2, \ldots$

Algorithm for 2-colouring

- **Messages:**
  - "ID x" = there is an endpoint with identifier x
  - "colour c" = my colour is c



Algorithm for 2-colouring

- **Messages:**
  - "ID *x*" = there is an endpoint with identifier *x*
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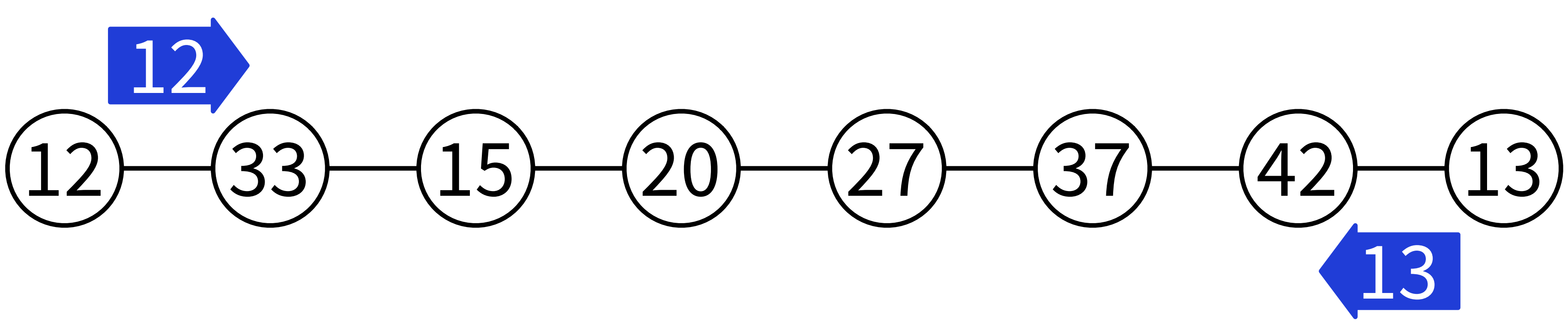

Algorithm for 2-colouring

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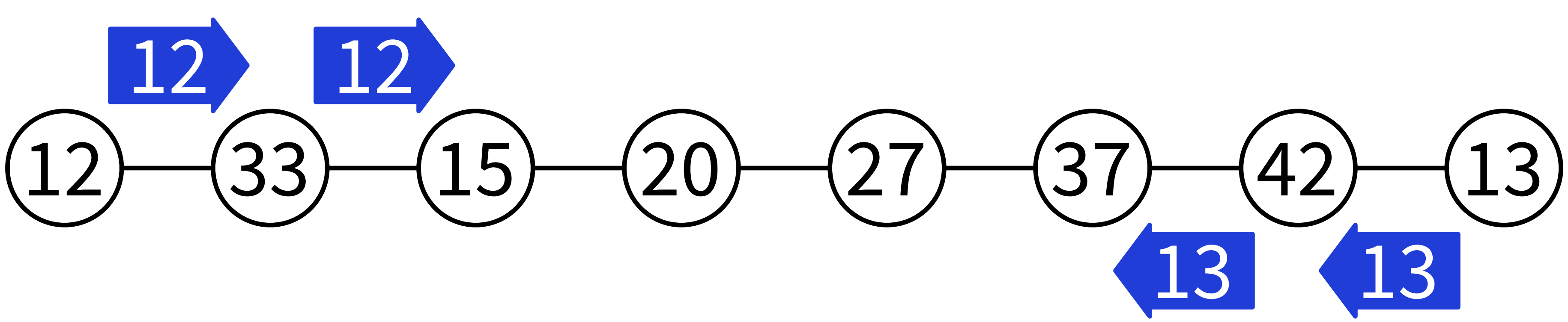

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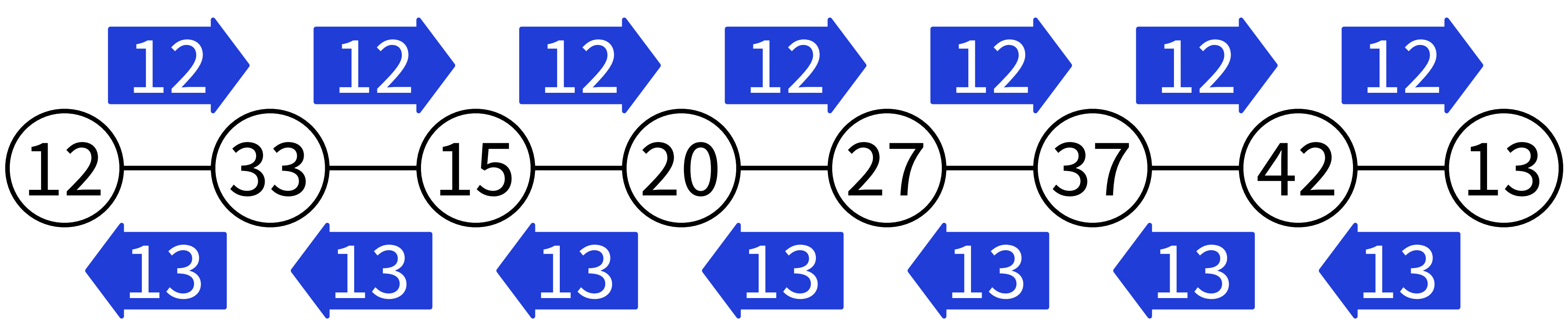

Algorithm for 2-colouring

- **Messages:**
  - "ID x" = there is an endpoint with identifier x
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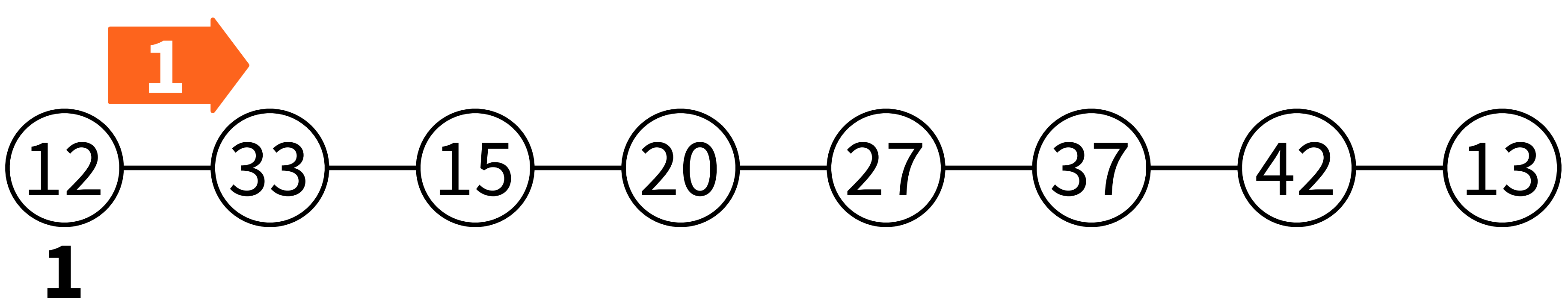
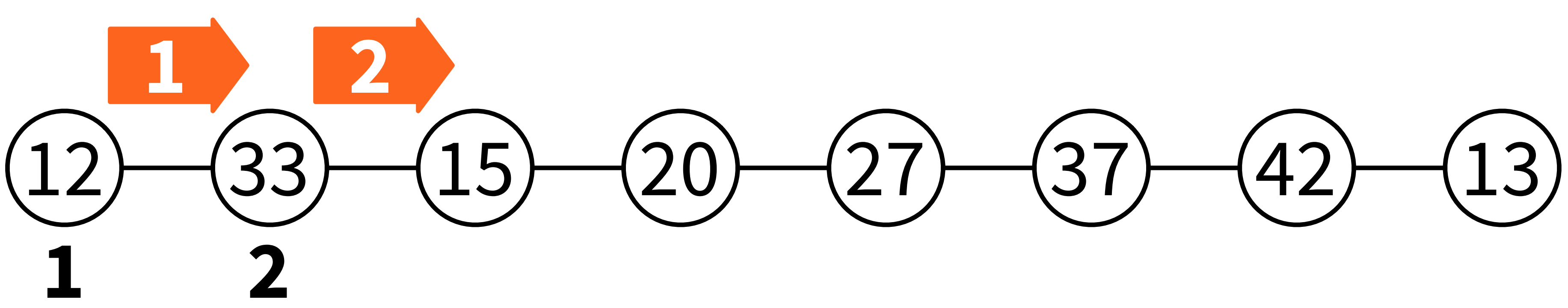


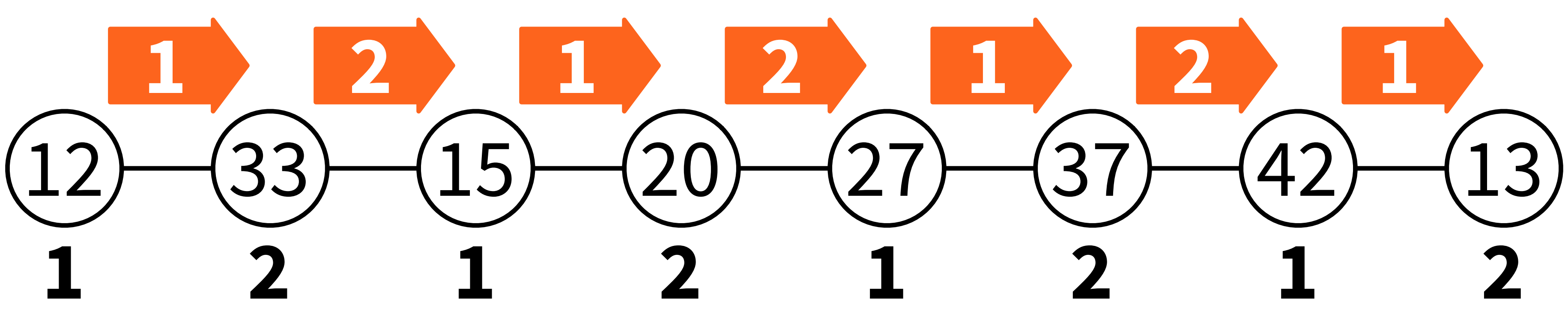
Algorithm for 2-colouring

- **Messages:**
  - "ID *x*" = there is an endpoint with identifier *x*
  - "colour *c*" = my colour is *c*



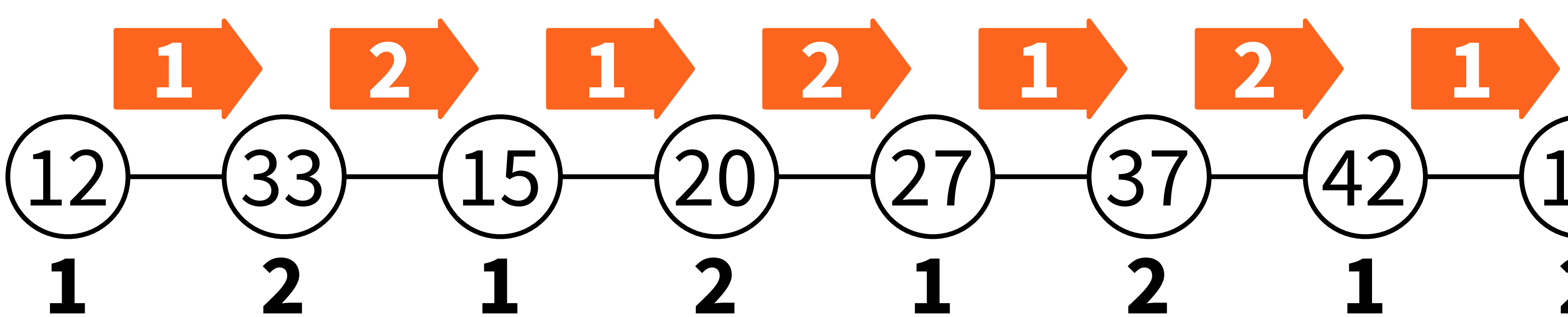
Algorithm for 2-colouring

- **Messages:**
  - “ID x” = there is an endpoint with identifier x
  - “colour c” = my colour is c



Algorithm for 2-colouring

- **States:** “I have have received ID x from left and next I will need to send it to right”, …
- **Running time:** $O(n)$ rounds



Algorithm for 2-colouring

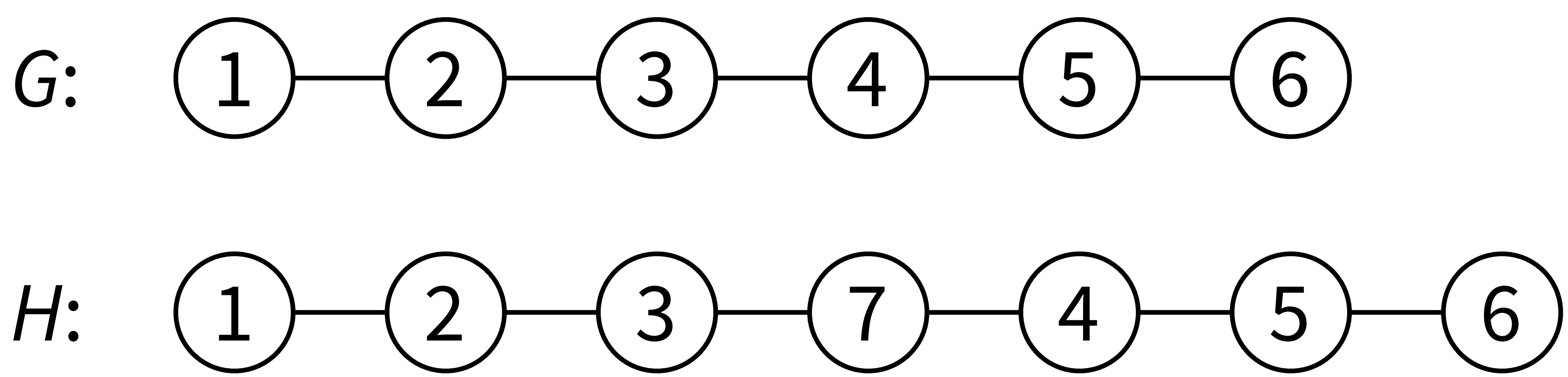
- **2-colouring possible in $O(n)$ rounds**
- **Goal: prove that this is optimal!**
  - there is no algorithm that finds a 2-colouring in time $o(n)$
  - assumptions: path, unique identifiers

Lower bound for 2-colouring

- Assume: there is an $o(n)$-time algorithm A
- For large n, running time << $n/2$
- Idea: construct two possible worlds, show that A fails in one of them

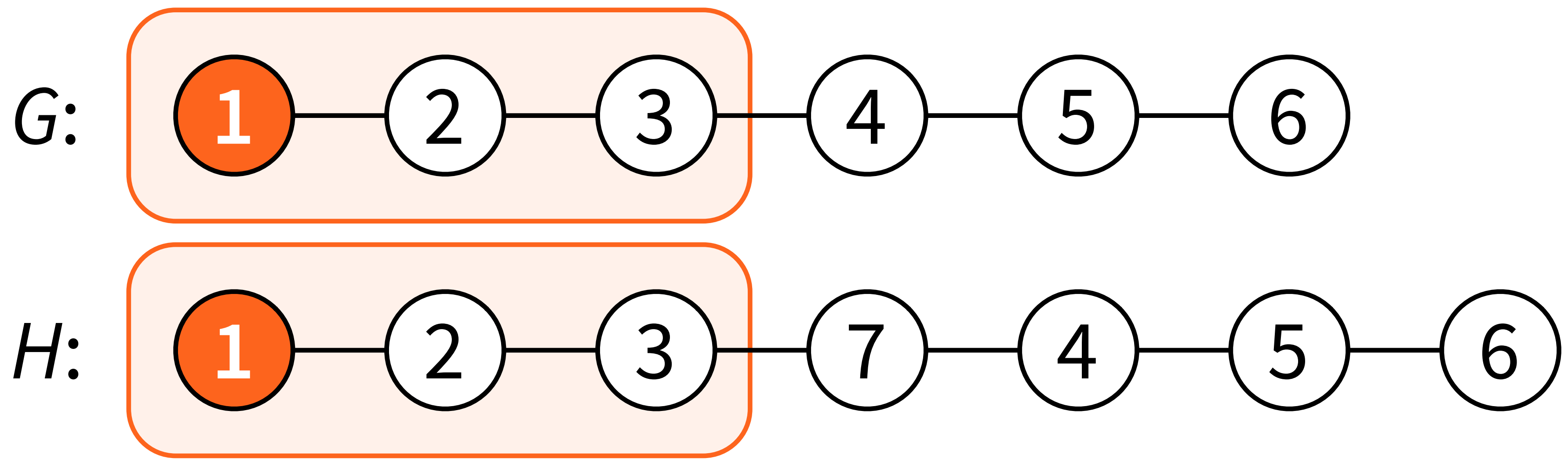
Lower bound for 2-colouring

- **Long paths with 2*k* and 2*k*+1 nodes, algorithm runs in ≤ *k*−1 rounds**



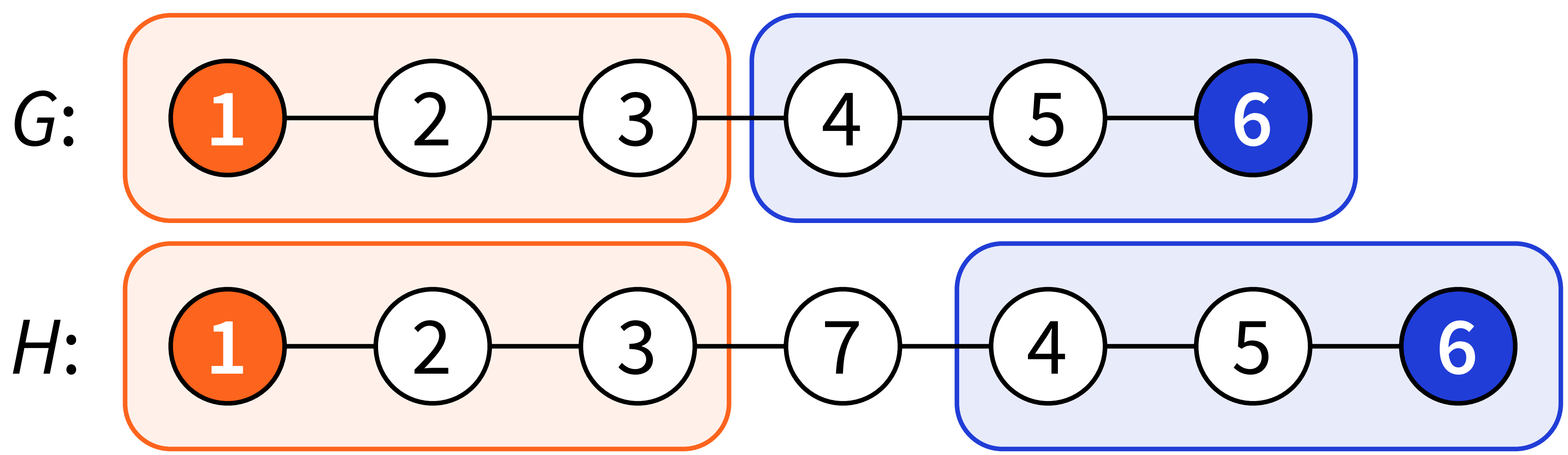
Lower bound for 2-colouring

- **Same (k−1)-neighbourhood, same output**



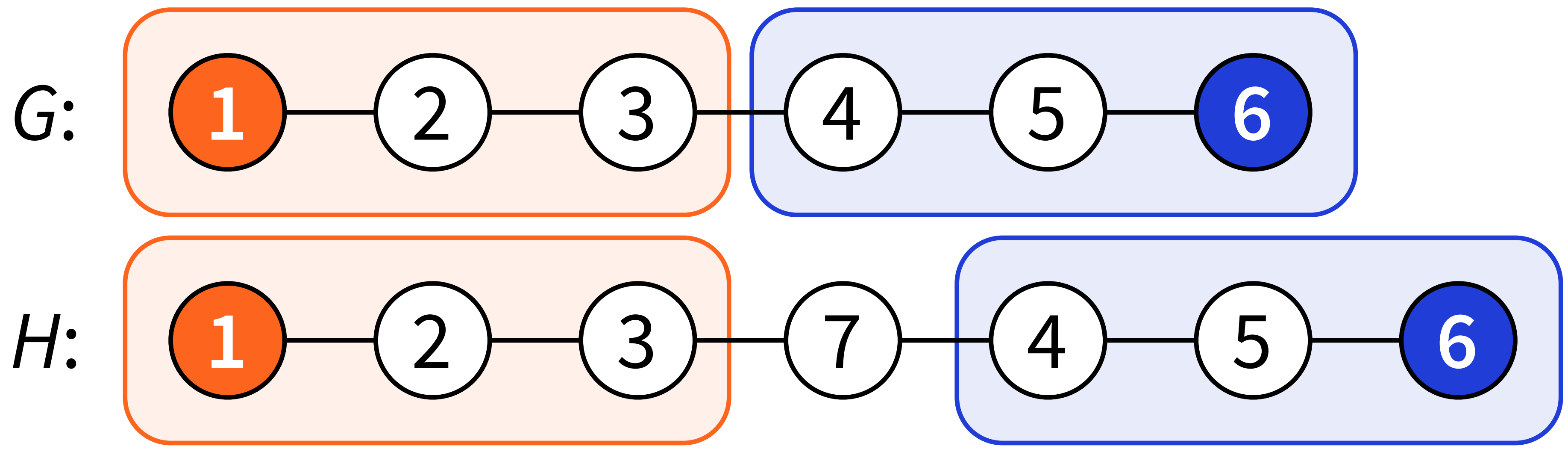
Lower bound for 2-colouring

- **Same (k−1)-neighbourhood, same output**



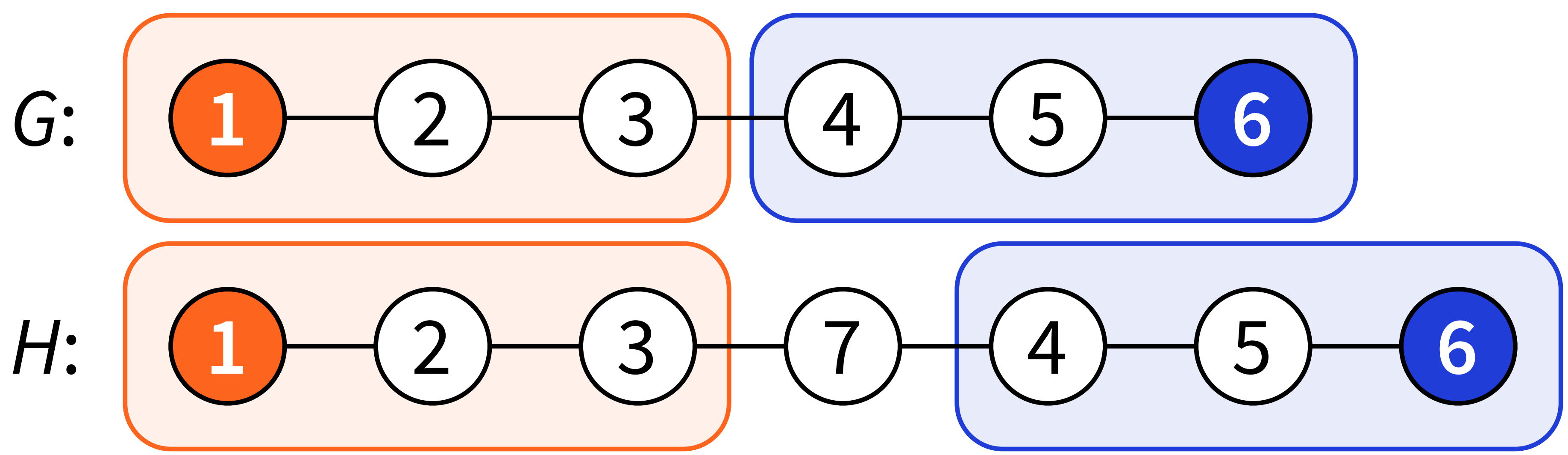
Lower bound for 2-colouring

- **Contradiction — why?**



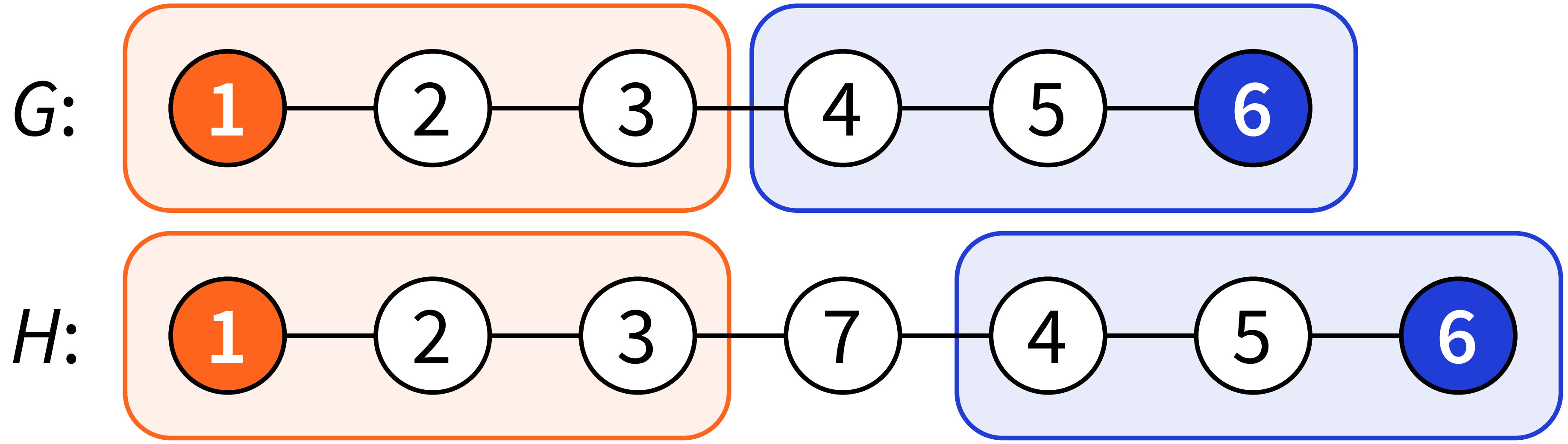
Lower bound for 2-colouring

- ***G***: nodes 1 and 6 must have different colours
- ***H***: nodes 1 and 6 must have the same colour



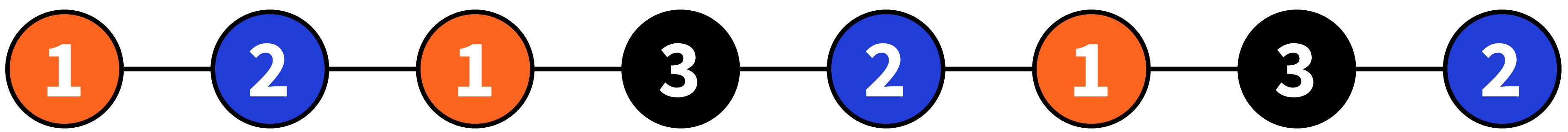
Lower bound for 2-colouring

- **Conclusion: there is no algorithm that finds a 2-colouring of a path in time $o(n)$**



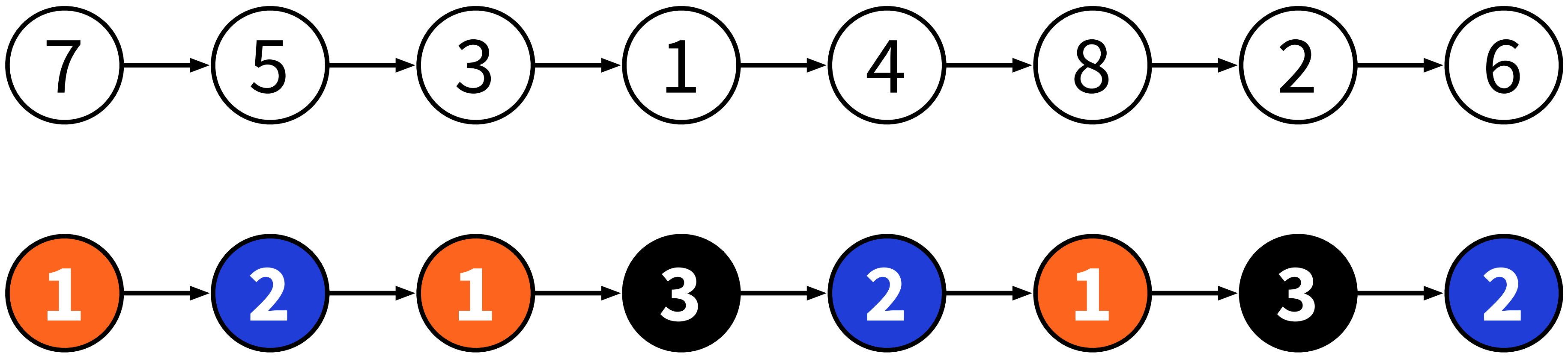
Using locality to prove lower bounds

- Example: 3-colouring of a path
- Upper bound: possible in time $O(\log^* n)$
- Lower bound: not possible in time $o(\log^* n)$



Lower bound for 3-colouring

- **Given: directed path with *n* nodes, identifiers are a permutation of {1, 2, …, *n*}**

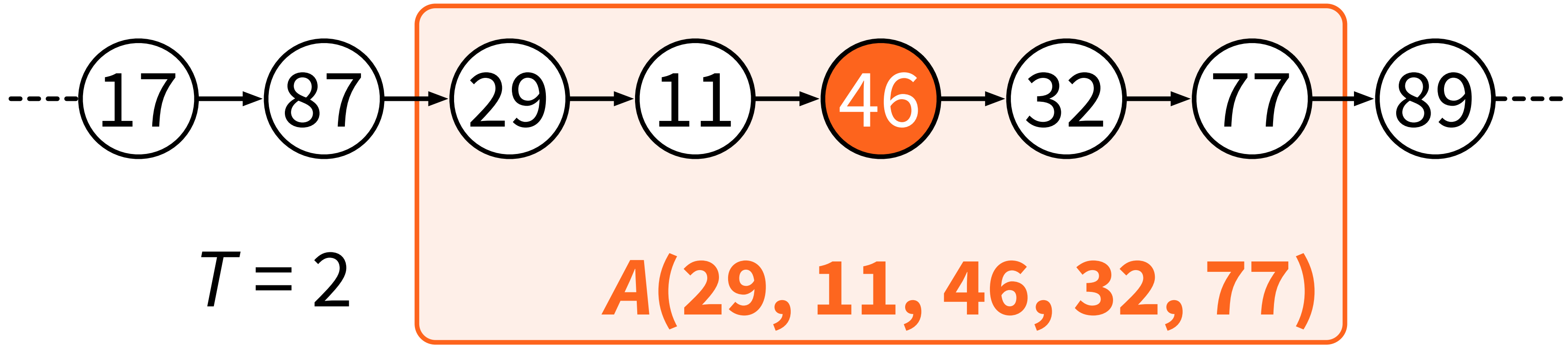


Lower bound for 3-colouring

- **Given: directed path with n nodes, identifiers are a permutation of $\{1, 2, \ldots, n\}$**
- **Assume: there is an algorithm A that finds a 3-colouring in time T**
- **Goal: prove that $T \geq \frac{1}{2} \log^*(n) - 1$**

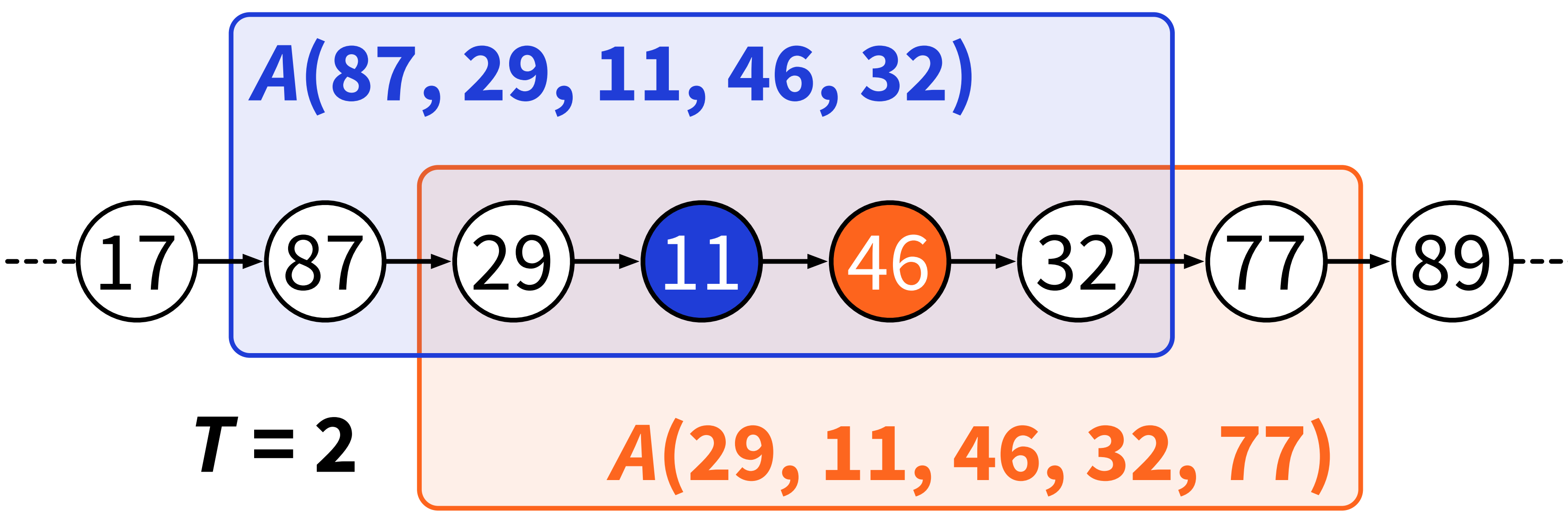
Algorithm for 3-colouring paths

- **Running time T = output only depends on radius-T neighbourhood of the node**
- **Algorithm = k-ary function where k = 2T+1**



Algorithm for 3-colouring paths

$$A(87, 29, 11, 46, 32) \neq A(29, 11, 46, 32, 77)$$



Algorithm for c-colouring paths

- $A(x_1, \ldots, x_k) \in \{1, \ldots, c\}$
  for all distinct $x_1, \ldots, x_k \in \{1, \ldots, n\}$

- $A(x_1, \ldots, x_k) \neq A(x_2, \ldots, x_{k+1})$
  for all distinct $x_1, \ldots, x_{k+1} \in \{1, \ldots, n\}$

Definition: **“*k*-ary *c*-colouring function”**

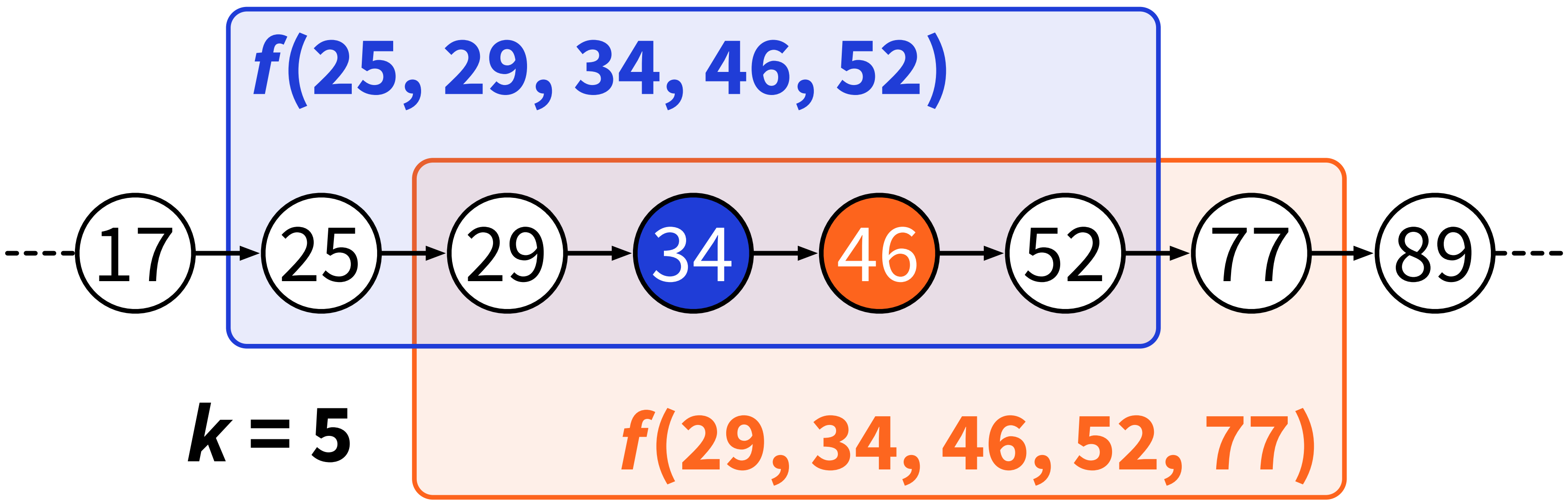
- $f(x_1, \ldots, x_k) \in \{1, \ldots, c\}$
  for all $1 \leq x_1 < \ldots < x_k \leq n$

- $f(x_1, \ldots, x_k) \neq f(x_2, \ldots, x_{k+1})$
  for all $1 \leq x_1 < \ldots < x_{k+1} \leq n$

- **We only care what happens with increasing identifiers**

k-ary *c*-colouring function

$$f(25, 29, 34, 46, 52) \neq f(29, 34, 46, 52, 77)$$



k-ary *c*-colouring function

- **Assume:** A is a distributed algorithm that finds a 3-colouring in directed n-cycles in time T
- **Then:** A is also a k-ary 3-colouring function for $k = 2T + 1$
- **Plan:** show that $k + 1 \geq \log^* n$

Lemma 1

- **If f is a 1-ary c-colouring function, then $c \geq n$**
- **Proof:**
  - pigeonhole principle
  - if $c < n$, there is a collision $f(x) = f(y)$ for some $1 \leq x < y \leq n$, contradiction

Lemma 2

- **If f is a k-ary c-colouring function, then we can construct a $(k-1)$-ary 2^c-colouring function g**
- **Proof:**
  - $g'(x_1, \ldots, x_{k-1}) = \{f(x_1, \ldots, x_{k-1}, y) : y > x_{k-1}\}$
  - $g(x_1, \ldots, x_{k-1}) = h(g'(x_1, \ldots, x_{k-1}))$
  - h = bijection that maps sets to colours

Lemma 2 (continued)

- $g'(x_1, \ldots, x_{k-1}) = \{f(x_1, \ldots, x_{k-1}, y) : y > x_{k-1}\}$
- $g(x_1,\ldots, x_{k-1}) = h(g'(x_1, \ldots, x_{k-1}))$
- h = bijection that maps sets to colours
- **By construction:** $g(x_1, \ldots, x_{k-1}) \in \{1, \ldots, 2^c\}$
- **Need to show:** $g(x_1, \ldots, x_{k-1}) \neq g(x_2, \ldots, x_k)$ for all $1 \leq x_1 < \ldots < x_k \leq n$

Lemma 2 (continued)

- $g'(x_1, \ldots, x_{k-1}) = \{f(x_1, \ldots, x_{k-1}, y) : y > x_{k-1}\}$
- $g(x_1,\ldots, x_{k-1}) = h(g'(x_1, \ldots, x_{k-1}))$
- h = bijection that maps sets to colours

- **Need to show:** $g(x_1, \ldots, x_{k-1}) \neq g(x_2, \ldots, x_k)$ for all $1 \leq x_1 < \ldots < x_k \leq n$

Lemma 2 (continued)

- $g'(x_1, \ldots, x_{k-1}) = \{f(x_1, \ldots, x_{k-1}, y) : y > x_{k-1}\}$
- $g(x_1,\ldots, x_{k-1}) = h(g'(x_1, \ldots, x_{k-1}))$
- h = bijection that maps sets to colours

- **Need to show:** $g'(x_1, \ldots, x_{k-1}) \neq g'(x_2, \ldots, x_k)$ for all $1 \leq x_1 < \ldots < x_k \leq n$

Lemma 2 (continued)

- $g'(x_1, \ldots, x_{k-1}) = \{f(x_1, \ldots, x_{k-1}, y) : y > x_{k-1}\}$

- **Need to show:** $g'(x_1, \ldots, x_{k-1}) \neq g'(x_2, \ldots, x_k)$ for all $1 \leq x_1 < \ldots < x_k \leq n$

Lemma 2 (continued)

- $1 \leq x_1 < x_2 < \ldots < x_k \leq n$
- $g'(x_1, \ldots, x_{k-1}) = \{f(x_1, \ldots, x_{k-1}, y) : y > x_{k-1}\}$
- $g'(x_2, \ldots, x_k) = \{f(x_2, \ldots, x_k, z) : z > x_k\}$
- $f(x_1, \ldots, x_{k-1}, x_k) \in g'(x_1, \ldots, x_{k-1})$
- $f(x_1, \ldots, x_{k-1}, x_k) \notin g'(x_2, \ldots, x_k)$
- $g'(x_1, \ldots, x_{k-1}) \neq g'(x_2, \ldots, x_k)$

Lemma 2 (continued)

- $1 \le x_1 < x_2 < \ldots < x_k \le n$
- $g'(x_1, \ldots, x_{k-1}) = \{f(x_1, \ldots, x_{k-1}, y) : y > x_{k-1}\}$
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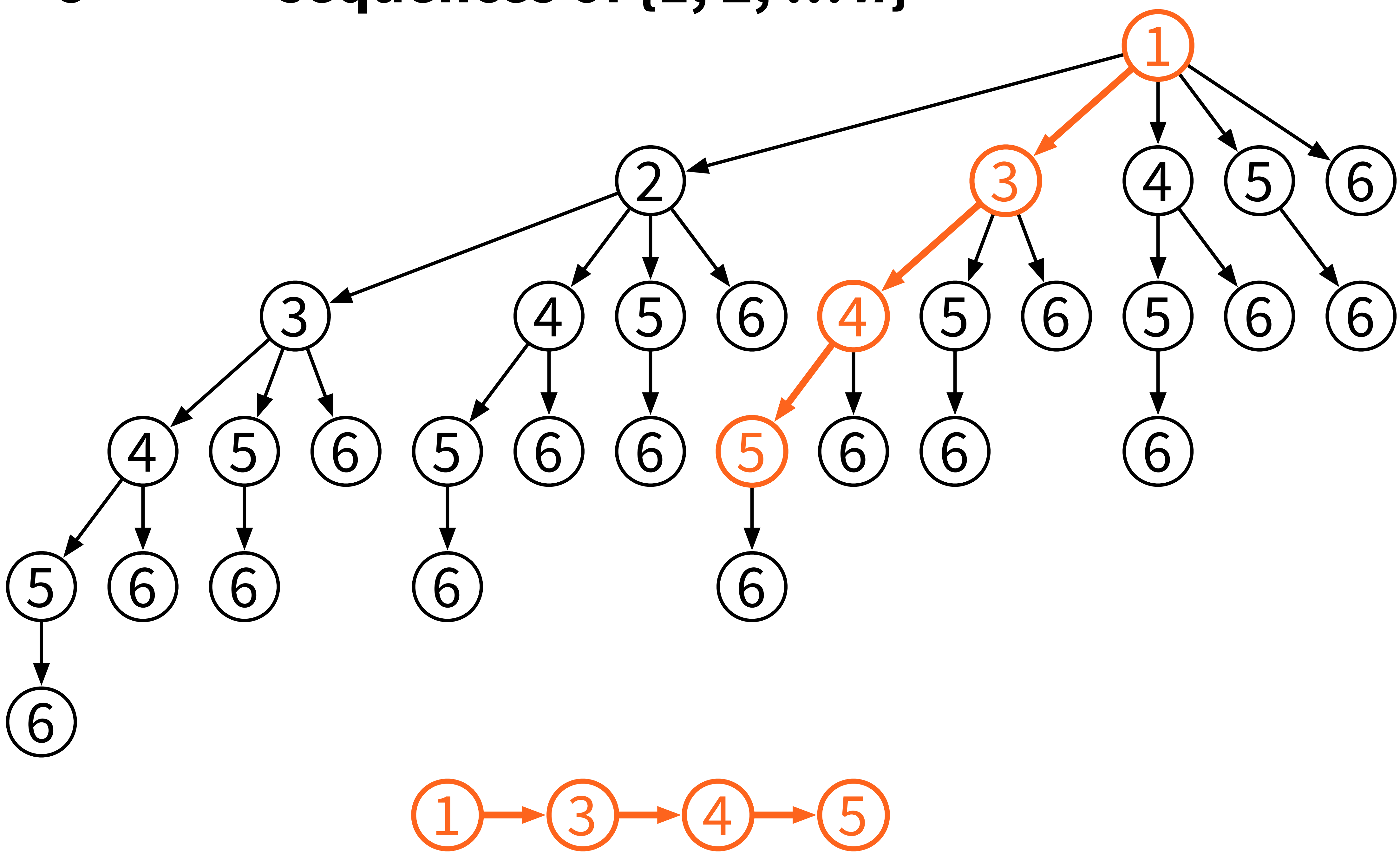
- $g(x_1, \ldots, x_{k-1}) \neq g(x_2, \ldots, x_k)$

$n = 6$

$k = 3$

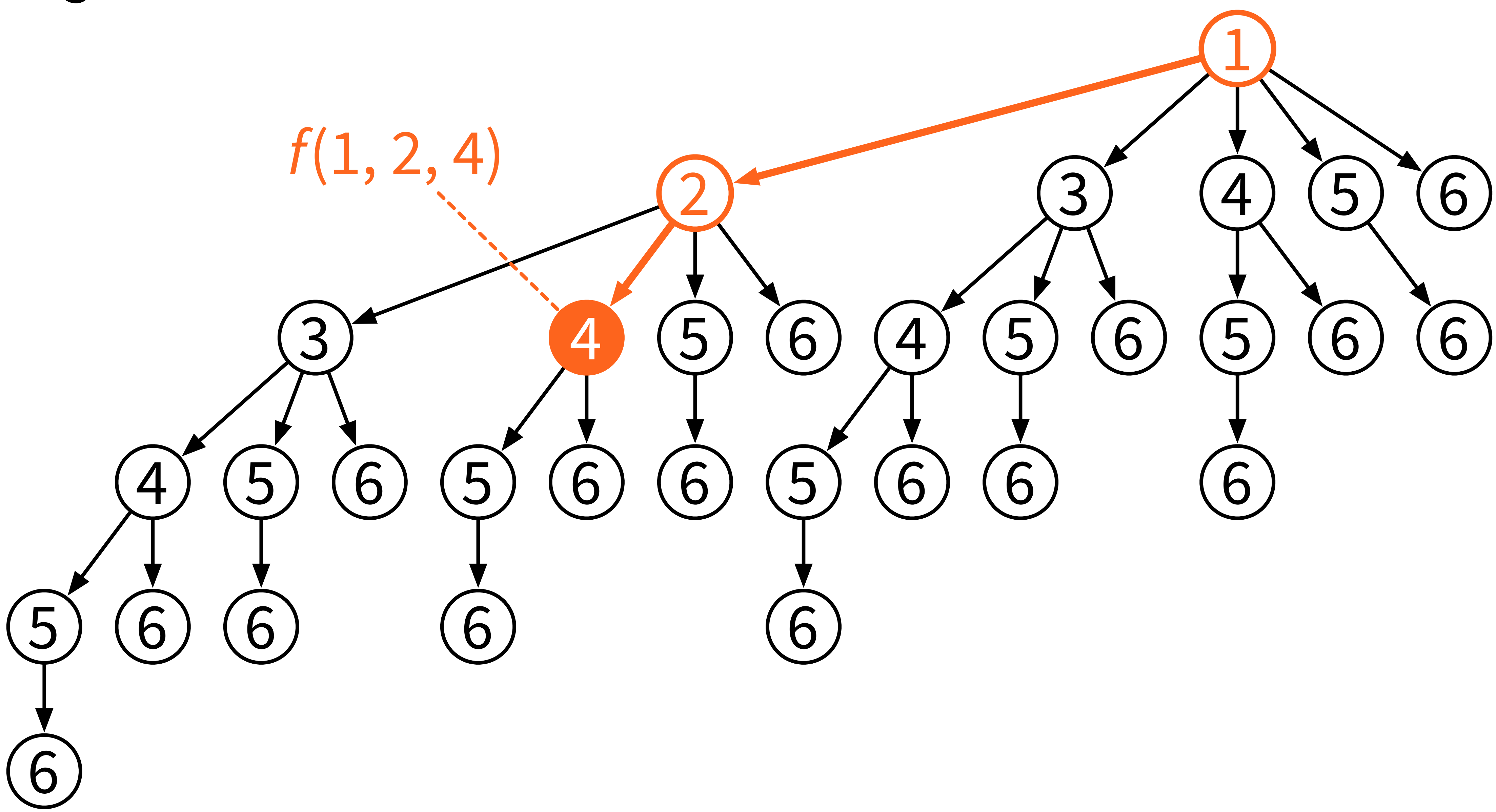
Tree that contains all increasing sequences of {1, 2, ... *n*}



$n = 6$

$k = 3$

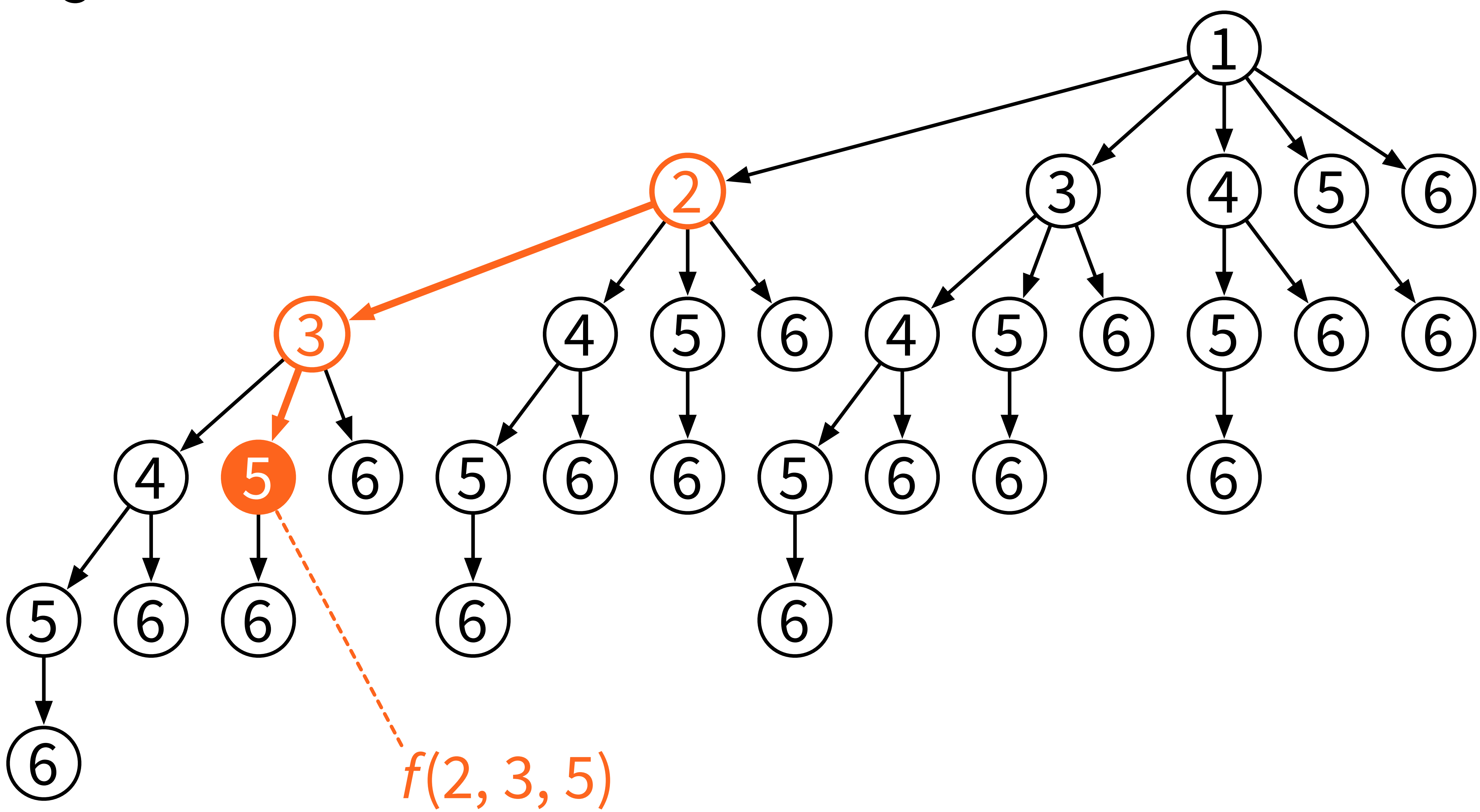
Colour of a node = value of *f*



$n = 6$

$k = 3$

Colour of a node = value of *f*

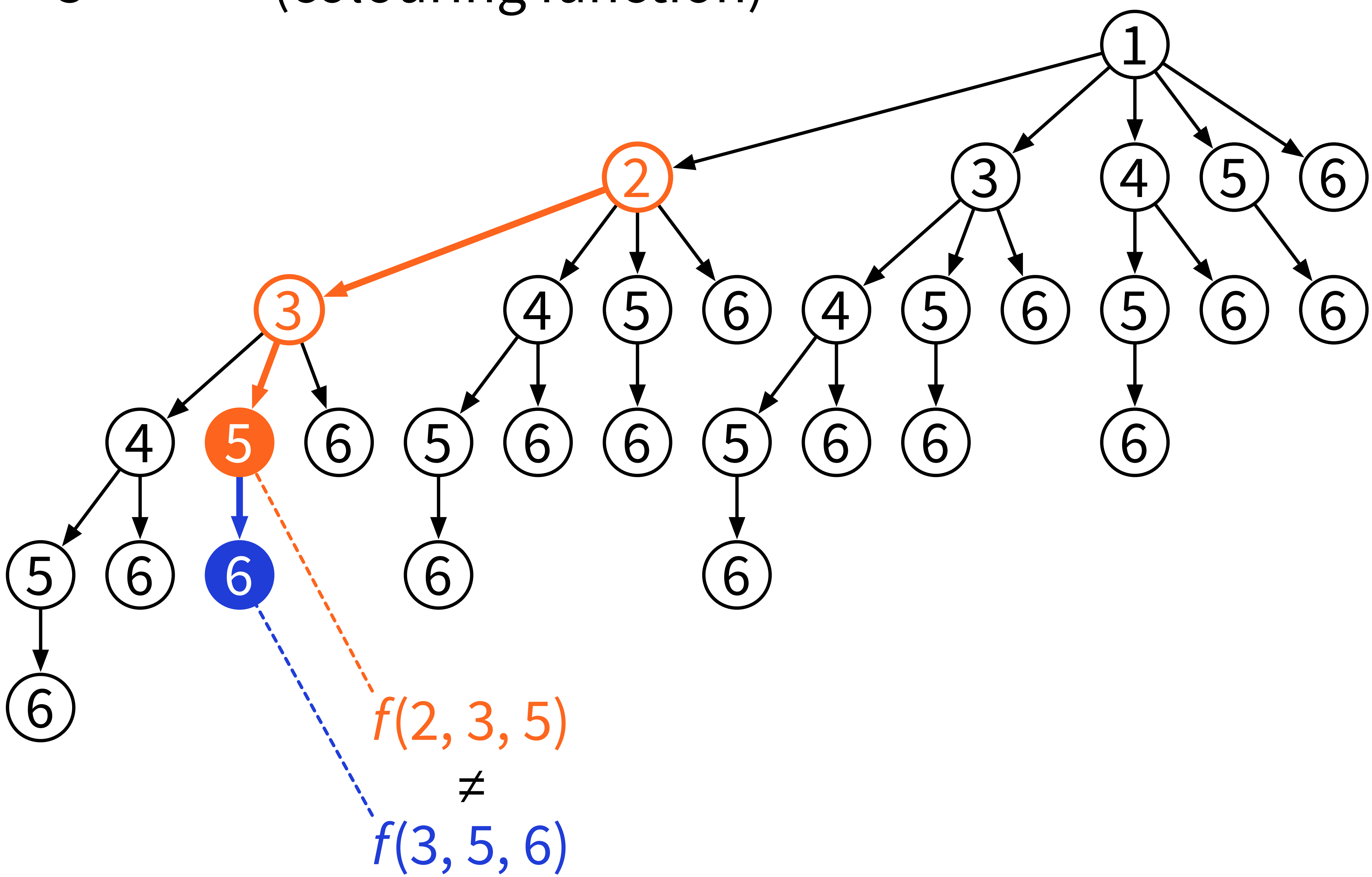


$n = 6$

$k = 3$

Colour of a node = value of *f*

(colouring function)



$n = 6$

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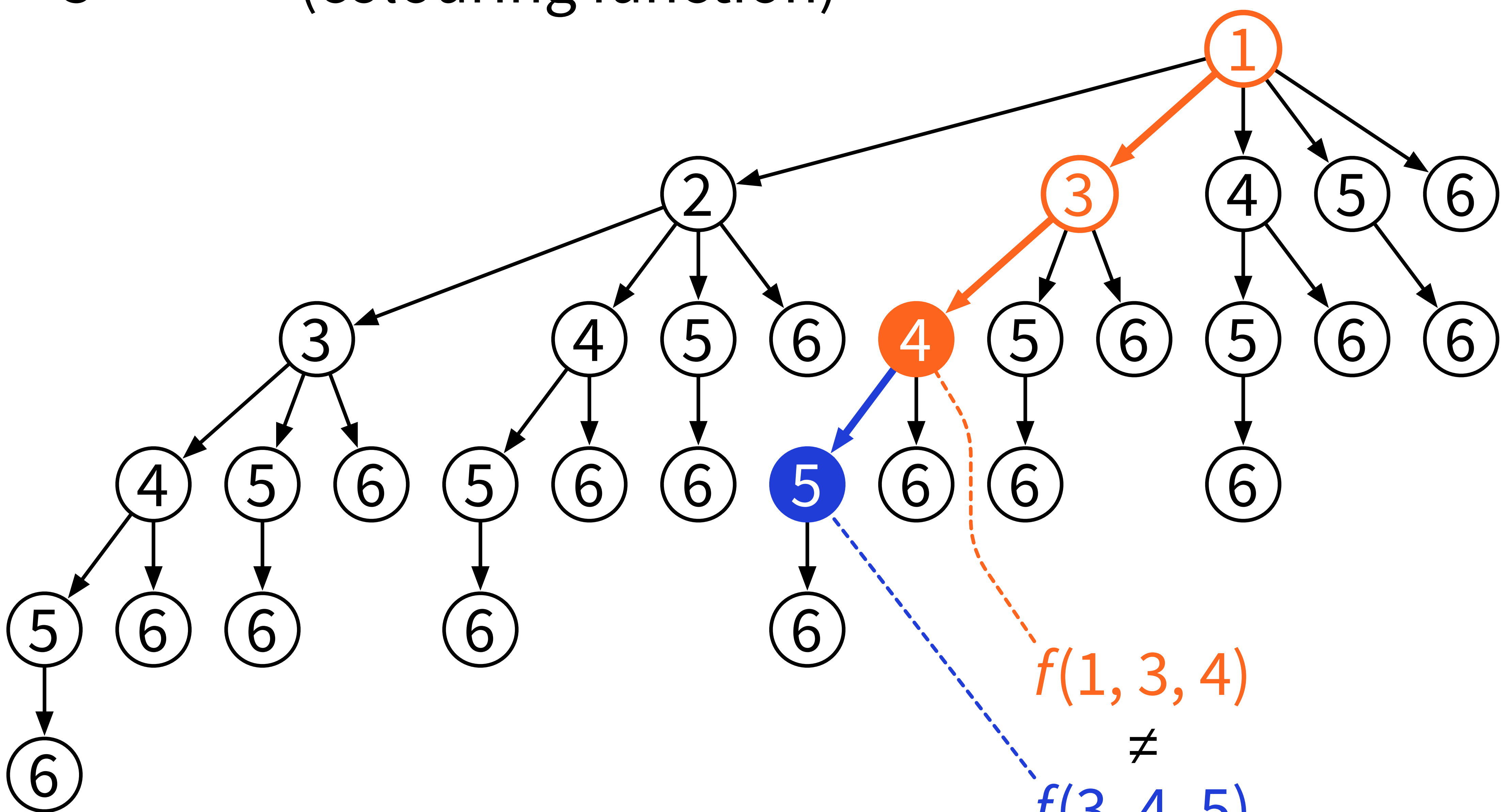
(colouring function)

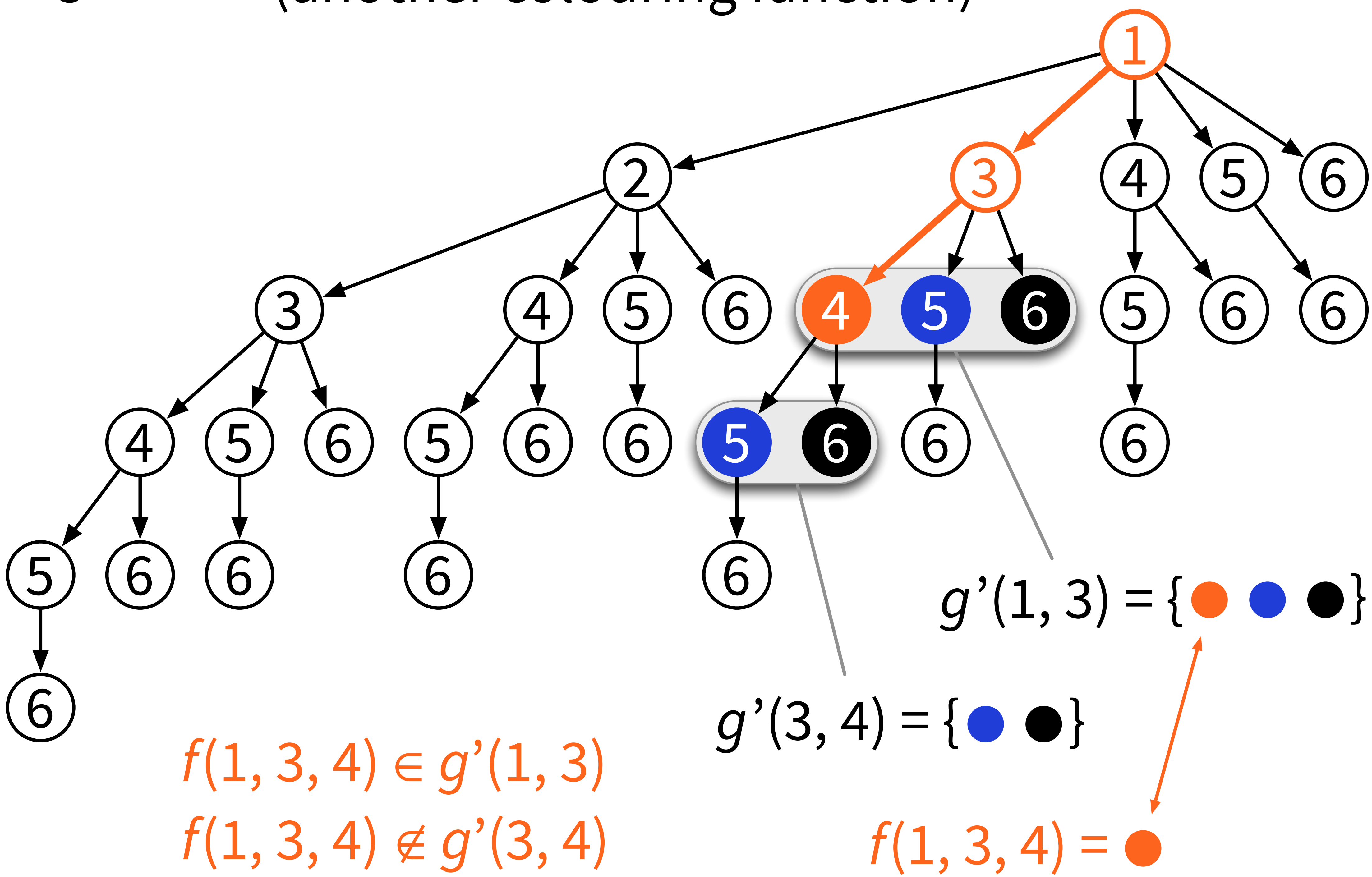


$n = 6$

$k = 3$

Colours of all children = value of *g*'

(another colouring function)

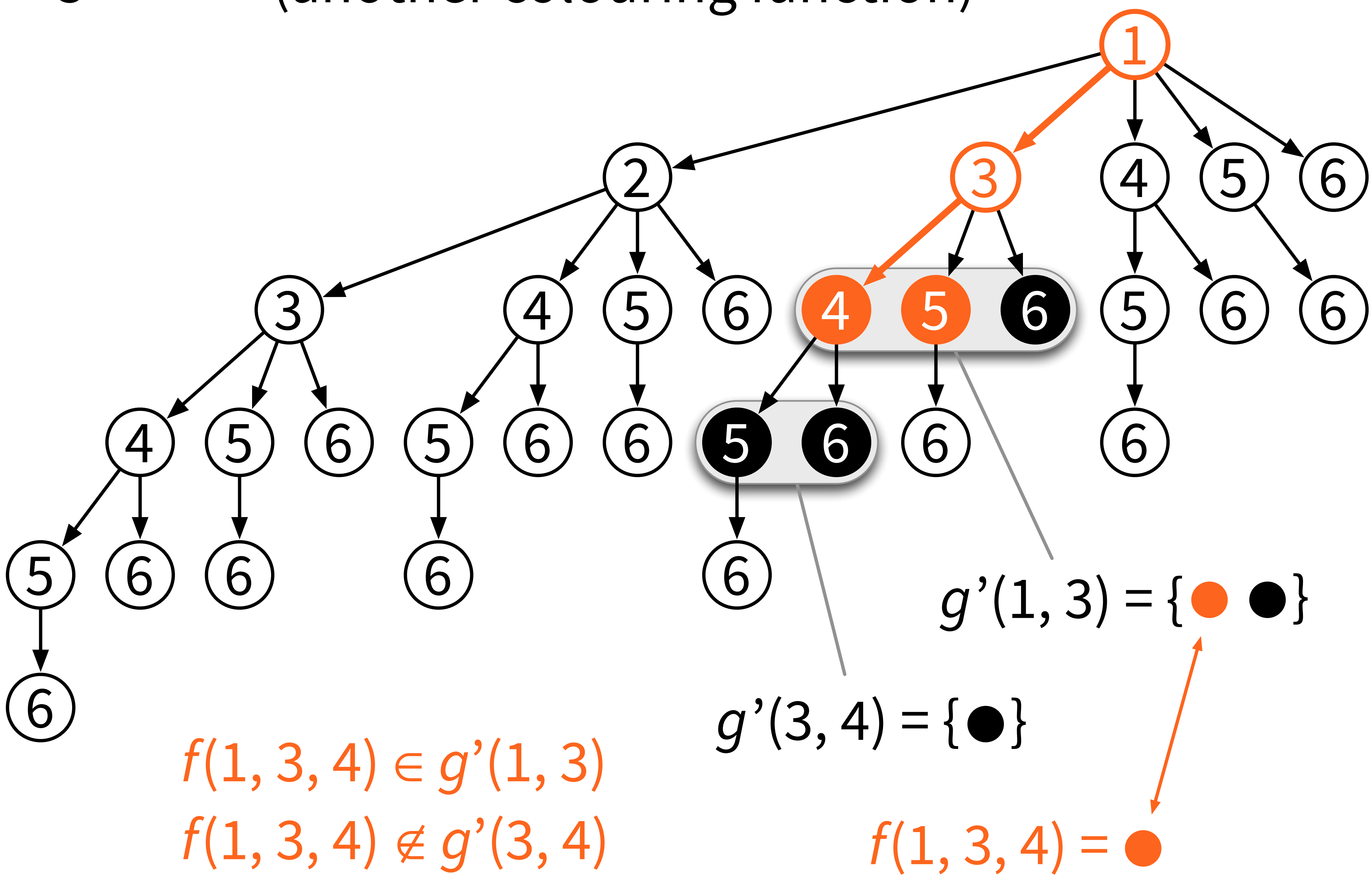


$n = 6$

$k = 3$

Colours of all children = value of *g*'

(another colouring function)

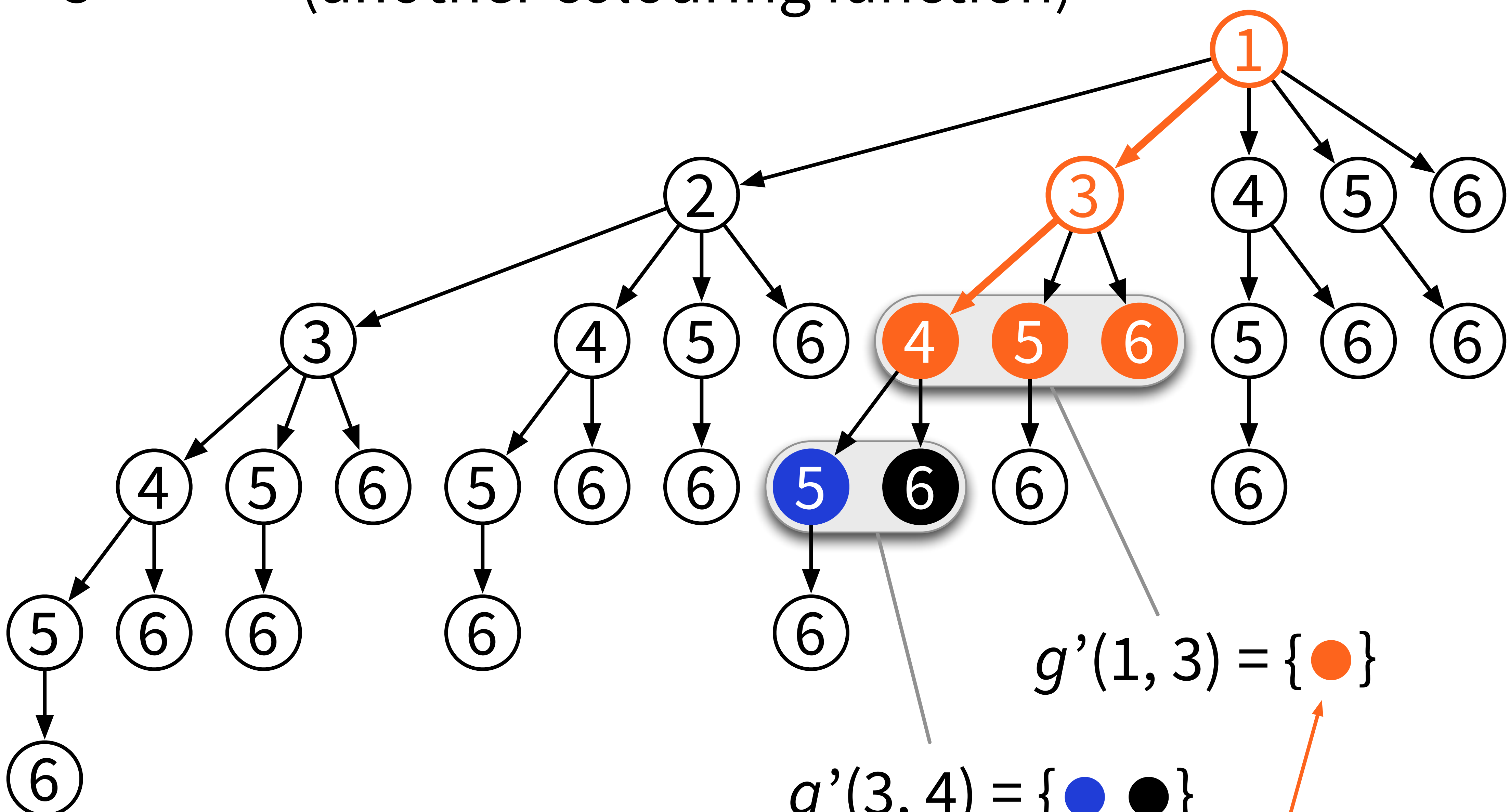


$n = 6$

$k = 3$

Colours of all children = value of *g'*

(another colouring function)



$f(1, 3, 4) \in g'(1, 3)$

$f(1, 3, 4) \notin g'(3, 4)$

$f(1, 3, 4) = \bullet$

Lemma 2

- **If f is a k-ary c-colouring function, then we can construct a $(k-1)$-ary 2^c-colouring function g**
- **Proof:**
  - $g'(x_1, \ldots, x_{k-1}) = \{f(x_1, \ldots, x_{k-1}, y) : y > x_{k-1}\}$
  - $g(x_1, \ldots, x_{k-1}) = h(g'(x_1, \ldots, x_{k-1}))$
  - h = bijection that maps sets to colours

Iterate Lemma 2

$^{i}2 = 2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}$ (i twos)

- k-ary 3-colouring function →
  k-ary $^{2}2$-colouring function →
  $(k-1)$-ary $^{3}2$-colouring function →
  $(k-2)$-ary $^{4}2$-colouring function →
  $(k-3)$-ary $^{5}2$-colouring function →
  …
  1-ary $^{k+1}2$-colouring function

Lemma 1 + Lemma 2

$${}^{i}2 = 2^{2^{\cdot^{\cdot^{\cdot^{2}}}}} \quad (i \text{ twos})$$

- **Lemma 2:**
  - k-ary 3-colouring function → 1-ary ${}^{k+1}2$-colouring function
- **Lemma 1:**
  - ${}^{k+1}2 \geq n$ (that is, $k + 1 \geq \log^* n$)

Lower bound for 3-colouring

- **Assume: A is a distributed algorithm that finds a 3-colouring in directed n-cycles in time T**
- **Then: A is also a k-ary 3-colouring function for $k = 2T + 1$**
- **Then: $k + 1 \geq \log^* n$, therefore: $T \geq \frac{1}{2} \log^*(n) - 1$**

Conclusions: tight bounds

- **2-colouring paths:**
  - possible in time $O(n)$
  - not possible in time $o(n)$
- **3-colouring paths:**
  - possible in time $O(\log^* n)$
  - not possible in time $o(\log^* n)$

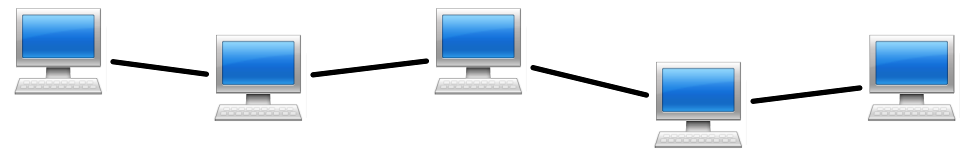
Assuming:
directed path,
unique IDs =
$\{1, 2, \ldots, n\}$

Conclusions: tight bounds

- **2-colouring paths:**
  - possible in time $O(n)$
  - not possible in time $o(n)$
- **3-colouring paths:**
  - possible in time $O(\log^* n)$
  - not possible in time $o(\log^* n)$

Richard Cole and Uzi Vishkin (1986)

Nathan Linial (1992)

- **Weeks 1–2: informal introduction**
  - network = path 
- **Week 3: graph theory**
- **Weeks 4–7: models of computing**
  - what can be computed (efficiently)?
- **Weeks 8–11: lower bounds**
  - what cannot be computed (efficiently)?
- **Week 12: recap**