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Distributed Algorithms 2020

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Chapter 8
Local Neighborhoods

Covering maps can be used to argue that a given problem cannot be solved *at all* with deterministic PN algorithms. Now we will study the concept of *locality*, which can be used to argue that a given problem cannot be solved *fast*, in any model of distributed computing.

8.1 Definitions

Let $N = (V, P, p)$ and $N' = (V', P', p')$ be simple port-numbered networks, with the underlying graphs $G = (V, E)$ and $G' = (V', E')$. Fix the local inputs $f : V \to Y$ and $f' : V' \to Y$, a pair of nodes $v \in V$ and $v' \in V'$, and a radius $r \in \mathbb{N}$. Define the radius-r neighborhoods

$$U = \operatorname{ball}_G(v, r), \quad U' = \operatorname{ball}_{G'}(v', r).$$

We say that (N, f, v) and (N', f', v') have *isomorphic radius-r neighborhoods* if there is a bijection $\psi : U \to U'$ with $\psi(v) = v'$ such that

(a) ψ preserves degrees: $\deg_N(v) = \deg_{N'}(\psi(v))$ for all $v \in U$.

(b) ψ preserves connections and port numbers: $p(u, i) = (v, j)$ if and only if $p'(\psi(u), i) = (\psi(v), j)$ for all $u, v \in U$.

(c) ψ preserves local inputs: $f(v) = f'(\psi(v))$ for all $v \in U$.

The function ψ is called an *r-neighborhood isomorphism from* (N, f, v) *to* (N', f', v'). See Figure 8.1 for an example.

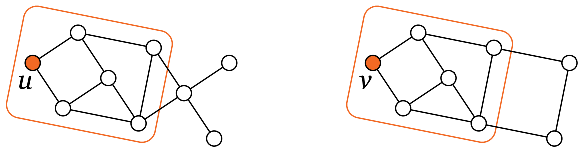


Figure 8.1: Nodes u and v have isomorphic radius-2 neighborhoods, provided that we choose the port numbers appropriately. Therefore in any algorithm A the state of u equals the state of v at time $t = 0, 1, 2$. However, at time $t = 3, 4, \dots$ this does not necessarily hold.

8.2 Local Neighborhoods and Executions

Theorem 8.1. *Assume that*

(a) A *is a deterministic* PN *algorithm with* $X = \mathrm{Input}_A$,

(b) $N = (V, P, p)$ *and* $N' = (V', P', p')$ *are simple port-numbered networks,*

(c) $f : V \to X$ *and* $f' : V' \to X$ *are arbitrary functions,*

(d) $v \in V$ *and* $v' \in V'$,

(e) (N, f, v) *and* (N', f', v') *have isomorphic radius-*r *neighborhoods.*

Let

(f) $x_0, x_1, \dots$ *be the execution of* A *on* (N, f), *and*

(g) $x'_0, x'_1, \dots$ *be the execution of* A *on* (N', f').

Then for each $t = 0, 1, \dots, r$ *we have* $x_t(v) = x'_t(v')$.

Proof. Let G and G' be the underlying graphs of N and N', respectively. We will prove the following stronger claim by induction: for each $t = 0, 1, \dots, r$, we have $x_t(u) = x'_t(\psi(u))$ for all $u \in \mathrm{ball}_G(v, r - t)$.

To prove the base case $t = 0$, let $u \in \mathrm{ball}_G(v, r)$, $d = \deg_N(u)$, and $u' = \psi(u)$; we have

$$x'_0(u') = \mathrm{init}_{A,d}(f'(u')) = \mathrm{init}_{A,d}(f(u)) = x_0(u).$$

For the inductive step, assume that $t \geq 1$ and

$$u \in \mathrm{ball}_G(v, r-t).$$

Let $u' = \psi(u)$. By inductive assumption, we have

$$x'_{t-1}(u') = x_{t-1}(u).$$

Now consider a port $(u,i) \in P$. Let $(s,j) = p(u,i)$. We have $\{s,u\} \in E$, and therefore

$$\mathrm{dist}_G(s,v) \leq \mathrm{dist}_G(s,u) + \mathrm{dist}_G(u,v) \leq 1 + r - t.$$

Define $s' = \psi(s)$. By inductive assumption we have

$$x'_{t-1}(s') = x_{t-1}(s).$$

The neighborhood isomorphism ψ preserves the port numbers: $(s',j) = p'(u',i)$. Hence all of the following are equal:

(a) the message sent by s to port j on round t,
(b) the message sent by s' to port j on round t,
(c) the message received by u from port i on round t,
(d) the message received by u' from port i on round t.

As the same holds for any port of u, we conclude that

$$x'_t(u') = x_t(u). \qquad \square$$

To apply Theorem 8.1 in the LOCAL model, we need to include unique identifiers in the local inputs f and f'.

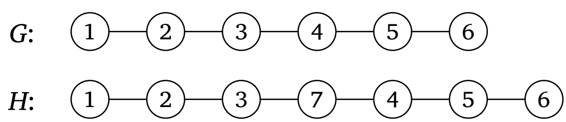
8.3 Example: 2-Coloring Paths

We know from Chapter 1 that one can find a proper 3-coloring of a path very fast, in $O(\log^* n)$ rounds. Now we will show that 2-coloring is much harder; it requires linear time.

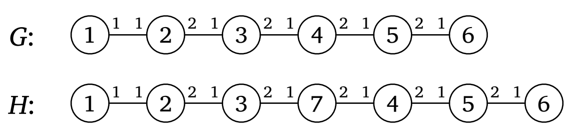
To reach a contradiction, suppose that there is a deterministic distributed algorithm A that finds a proper 2-coloring of any path graph in $o(n)$ rounds in the LOCAL model. Then there has to be a number n_0 such that for any number of nodes $n \geq n_0$, the running time of algorithm A is at most $(n-3)/2$. Pick some integer $k \geq n_0/2$, and consider two paths: path G contains $2k$ nodes, with unique identifiers $1, 2, \dots, 2k$, and path H contains $2k+1$ nodes, with unique identifiers

$$1, 2, \dots, k, 2k+1, k+1, k+2, \dots, 2k.$$

Here is an example for $k = 3$:



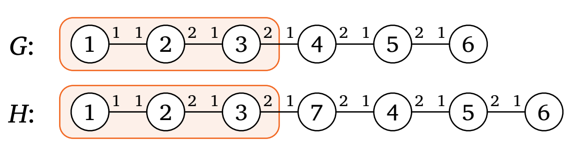
We assign the port numbers so that for all degree-2 nodes port number 1 points towards node 1:



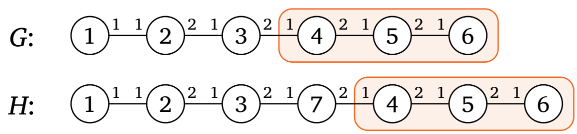
By assumption, the running time of A is at most

$$(n-3)/2 \leq (2k+1-3)/2 = k-1$$

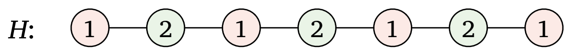
rounds in both cases. Since node 1 has got the same radius-$(k-1)$ neighborhood in G and H, algorithm A will produce the same output for node 1 in both networks:



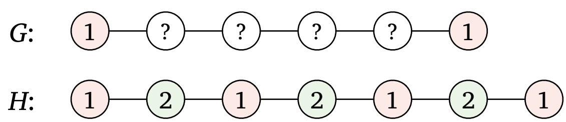
By a similar reasoning, node $2k$ (i.e., the last node of the path) also has to produces the same output in both cases:



However, now we reach a contradiction. In path H, in any proper 2-coloring nodes 1 and $2k$ have the same color—for example, both of them are of color 1, as shown in the following picture:



If algorithm A works correctly, it follows that nodes 1 and $2k$ must produce the same output in path H. However, then it follows that nodes 1 and $2k$ produces the same output also in G, too, but this cannot happen in any proper 2-coloring of G.



We conclude that algorithm A fails to find a proper 2-coloring in at least one of these instances.

8.4 Quiz

The caption of Figure 8.1 says: "However, at time $t = 3, 4, \ldots$ this does not necessarily hold." Design some deterministic algorithm A that demonstrates this for the PN model. When you run algorithm A in the network of Figure 8.1, the following should happen:

- At time $t = 0, 1, 2$, nodes u and v have the same local state.
- At time $t = 3, 4, \ldots$, nodes u and v have different local states.

Please try to design your algorithm so that its behavior does not depend on the choice of the port numbering: for each node x and each time step t, the state of x has to be the same regardless of how you choose the port numbers. A short, informal description of the algorithm is enough.

8.5 Exercises

Exercise 8.1 (edge coloring)**.** In this exercise, the graph family $\mathscr{F}$ consists of *path graphs*.

(a) Show that it is possible to find a 2-edge coloring in time $O(n)$ with deterministic PN-algorithms.

(b) Show that it is not possible to find a 2-edge coloring in time $o(n)$ with deterministic PN-algorithms.

(c) Show that it is not possible to find a 2-edge coloring in time $o(n)$ with deterministic LOCAL-algorithms.

Exercise 8.2 (maximal matching)**.** In this exercise, the graph family $\mathscr{F}$ consists of *path graphs*.

(a) Show that it is possible to find a maximal matching in time $O(n)$ with deterministic PN-algorithms.

(b) Show that it is not possible to find a maximal matching in time $o(n)$ with deterministic PN-algorithms.

(c) Show that it is possible to find a maximal matching in time $o(n)$ with deterministic LOCAL-algorithms.

Exercise 8.3 (optimization)**.** In this exercise, the graph family $\mathscr{F}$ consists of *path graphs*. Can we solve the following problems with deterministic PN-algorithms? If yes, how fast? Can we solve them any faster in the LOCAL model?

(a) Minimum vertex cover.

(b) Minimum dominating set.

(c) Minimum edge dominating set.

Exercise 8.4 (approximation). In this exercise, the graph family $\mathscr{F}$ consists of *path graphs*. Can we solve the following problems with deterministic PN-algorithms? If yes, how fast? Can we solve them any faster in the LOCAL model?

(a) 2-approximation of a minimum vertex cover?

(b) 2-approximation of a minimum dominating set?

Exercise 8.5 (auxiliary information). In this exercise, the graph family $\mathscr{F}$ consists of *path graphs*, and we are given a 4-coloring as input. We consider deterministic PN-algorithms.

(a) Show that it is possible to find a 3-coloring in time 1.

(b) Show that it is not possible to find a 3-coloring in time 0.

(c) Show that it is possible to find a 2-coloring in time $O(n)$.

(d) Show that it is not possible to find a 2-coloring in time $o(n)$.

⋆ **Exercise 8.6** (orientations). In this exercise, the graph family $\mathscr{F}$ consists of *cycle graphs*, and we are given some *orientation* as input. The task is to find a *consistent orientation*, i.e., an orientation such that both the indegree and the outdegree of each node is 1.

(a) Show that this problem cannot be solved with any deterministic PN-algorithm.

(b) Show that this problem cannot be solved with any deterministic LOCAL-algorithm in time $o(n)$.

(c) Show that this problem can be solved with a deterministic PN-algorithm if we give n as input to all nodes. How fast? Prove tight upper and lower bounds on the running time.

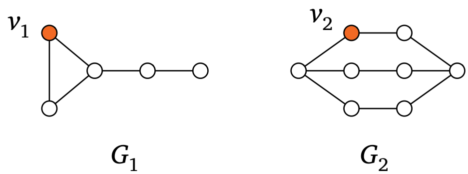


Figure 8.2: Graphs for Exercise 8.7.

⋆ **Exercise 8.7** (local indistinguishability)**.** Consider the graphs G_1 and G_2 illustrated in Figure 8.2. Assume that A is a deterministic PN-algorithm with running time 2. Show that A cannot distinguish between nodes v_1 and v_2. That is, there are simple port-numbered networks N_1 and N_2 such that N_i has G_i as the underlying graph, and the output of v_1 in N_1 equals the output of v_2 in N_2.

▹ *hint A*

8.6 Bibliographic Notes

Local neighborhoods were used to prove negative results in the context of distributed computing by, e.g., Linial [1].

8.7 Bibliography

[1] Nathan Linial. Locality in distributed graph algorithms. *SIAM Journal on Computing*, 21(1):193–201, 1992. doi:10.1137/0221015.

8.8 Hints

A. Argue using both covering maps and local neighborhoods. For $i = 1, 2$, construct a network N_i' and a covering map ϕ_i from N_i' to N_i. Let $v_i' \in \phi_i^{-1}(v_i)$. Show that v_1' and v_2' have isomorphic radius-2 neighborhoods; hence v_1' and v_2' produce the same output.

Then use the covering maps to argue that v_1 and v_2 also produce the same outputs. In the construction of N_1', you will need to eliminate the 3-cycle; otherwise v_1' and v_2' cannot have isomorphic neighborhoods.