Distributed Algorithms 2022

Hardness of coloring
This week’s goals

• **Specific technical result:**
  • 3-coloring of cycles in the LOCAL model
  • possible in $O(\log^* n)$ rounds (week 1)
  • not possible in $o(\log^* n)$ rounds (this week)

• **General idea:**
  • how to use round elimination to prove negative results in the **LOCAL** model and/or for **randomized** algorithms
Challenge & workaround

• Round elimination does not work directly in the **LOCAL** model
  • problem: **independence** vs. unique identifiers

• But we can use it to study **randomized** algorithms in the **PN** model
  • random bits are independent!

• Then results for the LOCAL model follow!
General idea:
Randomized round elimination
Randomized round elimination

• **The same pair of problems:** $X$ and $\text{re}(X)$
  • $\text{re}(X)$ does not depend on model of computing!

• **Different implications** in different models:
  • *if A is a deterministic PN-algorithm that solves $X$ in $T$ rounds then ...*
  • *if A is a randomized PN-algorithm that solves $X$ in $T$ rounds with high probability then ...*
Randomized round elimination

• We will use cycles as an example
• The same idea generalizes to biregular trees
  • probabilities that we get are just slightly different
Randomized round elimination in cycles

• $A_0$: local failure probability $< 1/x^3$  
  e.g. 0.1%

• $A_1$: form the set of frequent labels
  • labels that appear with probability $\geq 1/x$
  e.g. 10%

• Analysis: focus on lucky neighborhoods
  • neighborhoods in which $A_0$ fails with probability $< 1/x^2$
  e.g. 1%
Intuition

• Before seeing anything:
  • we know that $A_0$ failure rate is $< 1/x^3$

• Gather more local information:
  • gain more information on $A_0$ failure rate here
  • may increase or decrease — does it exceed $1/x^2$?
  • “unlucky”: much worse than average failure rate
  • “lucky”: not much worse than average failure rate
New active nodes

• Assume we are in a *lucky neighborhood*
  • by definition: \( P[A_0 \text{ fails}] < \frac{1}{x^2} \)

• Assume \([a, b]\) is a pair of *frequent labels*
  • happens here with probability \( \geq \frac{1}{x} \cdot \frac{1}{x} = \frac{1}{x^2} \)
  • \( A_0 \) cannot fail here with probability \( \geq \frac{1}{x^2} \)
  • **label pair \([a, b]\) must be feasible!**

• \( A_1 \) can fail only in unlucky neighborhoods!
Lucky neighborhoods

• **Assumption:** \( P[A_0 \text{ fails}] < 1/x^3 \)  
  e.g. 0.1%

• **Definition:** \( P[A_0 \text{ fails} | \text{ unlucky}] \geq 1/x^2 \)  
  e.g. 1%

• \( P[A_0 \text{ fails} | \text{ unlucky}] \cdot P[\text{ unlucky}] < 1/x^3 \)

• \( P[\text{ unlucky}] < 1/x \)  
  e.g. 10%
New passive nodes

- $P[A_0 \text{ fails}] < \frac{1}{x^3}$
- $P[A_0 \text{ output considered infrequent by } A_1] < \text{#labels} \cdot \text{#edges} \cdot \frac{1}{x}$

**Otherwise:**
- $A_0$ does not fail, its outputs form a valid solution
- $A_0$ outputs only labels that $A_1$ considers frequent
- $A_1$ has to succeed in solving $\text{re}(X)$
Summary

• $P[A_0 \text{ fails}] < \frac{1}{x^3}$

• Possible $A_1$-failures:
  • $P[\text{unlucky}] < \frac{1}{x}$
  • $P[A_0 \text{ fails}] < \frac{1}{x^3}$
  • $P[A_0 \text{ outputs some infrequent label}] < \#\text{labels} \cdot \#\text{edges} \cdot \frac{1}{x}$

• $P[A_1 \text{ fails}] < \text{constant} \cdot \frac{1}{x}$
Randomized round elimination in cycles

- $A_0$: local failure probability $< \frac{1}{x^3}$
- $A_1$: local failure probability $< \text{constant} \cdot \frac{1}{x}$
- Failure probability increases polynomially
- We can repeat this many times before $A_k$ becomes useless
What works very often

• Do round elimination in deterministic PN model
  • gain intuition on how the problem behaves
• Then switch to randomized PN model
  • proper analysis of failure probabilities
• Results for deterministic & randomized LOCAL follow directly
Case study: Coloring directed cycles
$n^{O(1)}$-coloring
0 rounds
(unique IDs)

3-coloring
$O(\log^* n)$ steps
$O(\log^* n)$ rounds

$2^k$-coloring
$T - 1$ rounds

2$^k$-coloring
$T$ rounds

$k$-coloring

Fast color reduction

Round elimination

$c$-coloring
0 rounds

3-coloring
$T \ll \log^* n$ rounds

$T$ steps
Sinkless orientation

• **Deterministic PN:**
  - not possible in $o(\log n)$ rounds  (last week)
  - possible in $O(\log n)$ rounds  (last week)

• **Randomized PN:**
  - not possible in $o(\log \log n)$ rounds  (exercise)
  - possible in $O(\log \log n)$ rounds  (not easy)

• **Deterministic LOCAL?**