Hardness of coloring
This week’s goals

**Specific technical result:**
- 3-coloring of cycles in the LOCAL model
- possible in $O(\log^* n)$ rounds (week 1)
- not possible in $o(\log^* n)$ rounds (this week)

**General idea:**
- how to use round elimination to prove negative results in the **LOCAL** model and/or for *randomized* algorithms
Challenge & workaround

• Round elimination does not work directly in the \textit{LOCAL} model
  • problem: \textit{independence} vs. unique identifiers
• But we can use it to study \textit{randomized} algorithms in the \textit{PN} model
  • random bits are independent!
• Then results for the LOCAL model follow!
General idea:
Randomized round elimination
Randomized round elimination

• **The same pair of problems**: $X$ and $\text{re}(X)$
  • $\text{re}(X)$ does not depend on model of computing!

• **Different implications** in different models:
  • *if A is a deterministic PN-algorithm that solves X in T rounds then ...*
  • *if A is a randomized PN-algorithm that solves X in T rounds with high probability then ...*
Randomized round elimination

• We will use cycles as an example

• The same idea generalizes to biregular trees
  • probabilities that we get are just slightly different
Randomized round elimination in cycles

- $A_0$: local failure probability $< 1/x^3$
  
- $A_1$: form the set of *frequent labels*
  
  - labels that appear with probability $\geq 1/x$

- **Analysis**: focus on *lucky neighborhoods*
  
  - neighborhoods in which $A_0$ fails with probability $< 1/x^2$

  - e.g. 0.1%
  - e.g. 10%
  - e.g. 1%
Intuition

• **Before seeing anything:**
  • we know that $A_0$ failure rate is $< 1/x^3$

• **Gather more local information:**
  • gain more information on $A_0$ failure rate here
  • may increase or decrease — does it exceed $1/x^2$?
  • “unlucky”: much worse than average failure rate
  • “lucky”: not much worse than average failure rate
New active nodes

• Assume we are in a **lucky neighborhood**
  • by definition: \( P[A_0 \text{ fails}] < \frac{1}{x^2} \)

• Assume \([a, b]\) is a pair of **frequent labels**
  • happens here with probability \( \geq \frac{1}{x} \cdot \frac{1}{x} = \frac{1}{x^2} \)
  • \(A_0\) cannot fail here with probability \( \geq \frac{1}{x^2} \)
  • **label pair** \([a, b]\) **must be feasible**!

• \(A_1\) can fail only in unlucky neighborhoods!
Lucky neighborhoods

• **Assumption:** $P[A_0 \text{ fails}] < 1/x^3$  
  e.g. 0.1%

• **Definition:** $P[A_0 \text{ fails | unlucky}] \geq 1/x^2$  
  e.g. 1%

• $P[A_0 \text{ fails | unlucky}] \cdot P[\text{unlucky}] < 1/x^3$

• $P[\text{unlucky}] < 1/x$  
  e.g. 10%
New passive nodes

• $\Pr[A_0 \text{ fails}] < \frac{1}{x^3}$

• $\Pr[A_0 \text{ output considered infrequent by } A_1] < \#\text{labels} \cdot \#\text{edges} \cdot \frac{1}{x}$

• Otherwise:
  • $A_0$ does not fail, its outputs form a valid solution
  • $A_0$ outputs only labels that $A_1$ considers frequent
  • $A_1$ has to succeed in solving $\text{re}(X)$
Summary

• $P[A_0 \text{ fails}] < \frac{1}{x^3}$

• Possible $A_1$-failures:
  • $P[\text{unlucky}] < \frac{1}{x}$
  • $P[A_0 \text{ fails}] < \frac{1}{x^3}$
  • $P[A_0 \text{ outputs some infrequent label}] < #\text{labels} \cdot #\text{edges} \cdot \frac{1}{x}$

• $P[A_1 \text{ fails}] < \text{constant} \cdot \frac{1}{x}$
Randomized round elimination in cycles

• $A_0$: local failure probability $< \frac{1}{x^3}$
• $A_1$: local failure probability $< \text{constant} \cdot \frac{1}{x}$
• Failure probability increases polynomially
• We can repeat this many times before $A_k$ becomes useless
What works very often

• Do round elimination in deterministic PN model
  • gain intuition on how the problem behaves

• Then switch to randomized PN model
  • proper analysis of failure probabilities

• Results for deterministic & randomized LOCAL
  follow directly
Case study: **Coloring directed cycles**
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<th>Column 3</th>
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Sinkless orientation

- **Deterministic PN:**
  - not possible in $o(\log n)$ rounds (last week)
  - possible in $O(\log n)$ rounds (last week)

- **Randomized PN:**
  - not possible in $o(\log \log n)$ rounds (exercise)
  - possible in $O(\log \log n)$ rounds (not easy)

- **Deterministic LOCAL?**