Aalto University

Distributed Algorithms 2024

11 Hardness of coloring

Today's goals

- **Specific technical result:**
  - 3-coloring of cycles in the LOCAL model
  - possible in **$O(\log^* n)$** rounds (week 1)
  - not possible in **$o(\log^* n)$** rounds (this week)
- **General idea:**
  - how to use round elimination to prove negative results in the ***LOCAL*** model and/or for ***randomized*** algorithms

Challenge & workaround

- Round elimination does not work directly in the *LOCAL* model
  - problem: **independence** vs. unique identifiers
- But we can use it to study ***randomized*** algorithms in the ***PN*** model
  - random bits are independent!
- Then results for the LOCAL model follow!

General idea:

Randomized round elimination

Randomized round elimination

- **The same pair of problems:** X and re(X)
  - re(X) does not depend on model of computing!
- **Different implications** in different models:
  - *if A is a deterministic PN-algorithm that solves X in T rounds then …*
  - ***if A is a randomized PN-algorithm that solves X in T rounds with high probability then …***

Randomized round elimination

- We will use cycles as an example
- The same idea generalizes to biregular trees
  - probabilities that we get are just slightly different

Randomized round elimination in cycles

- ***A_0*:** local failure probability $< 1/x^3$ — e.g. 0.1%
- ***A_1*:** form the set of ***frequent labels***
  - labels that appear with probability $\geq 1/x$ — e.g. 10%
- **Analysis:** focus on ***lucky neighborhoods***
  - neighborhoods in which A_0 fails with probability $< 1/x^2$ — e.g. 1%

Intuition

- **Before seeing anything:**
  - we know that A_0 failure rate is $< 1/x^3$
- **Gather more local information:**
  - gain more information on A_0 failure rate here
  - may increase or decrease — does it exceed $1/x^2$?
  - "unlucky": much worse than average failure rate
  - "lucky": not much worse than average failure rate

New active nodes

- Assume we are in a ***lucky neighborhood***
  - by definition: P[A_0 fails] < $1/x^2$
- Assume [a, b] is a pair of ***frequent labels***
  - happens here with probability ≥ $1/x \cdot 1/x = 1/x^2$
  - A_0 cannot fail here with probability ≥ $1/x^2$
  - **label pair [*a*, *b*] must be feasible!**
- A_1 can fail only in unlucky neighborhoods!

Lucky neighborhoods

- **Assumption:** P[A_0 fails] < $1/x^3$ — e.g. 0.1%
- **Definition:** P[A_0 fails | unlucky] ≥ $1/x^2$ — e.g. 1%
- P[A_0 fails | unlucky] · P[unlucky] < $1/x^3$
- P[unlucky] < **1/x** — e.g. 10%

New passive nodes

- $P[A_0 \text{ fails}] <$ **$1/x^3$**
- $P[A_0 \text{ output considered infrequent by } A_1]$
  $<$ **#labels · #edges · $1/x$**
- **Otherwise:**
  - A_0 does not fail, its outputs form a valid solution
  - A_0 outputs only labels that A_1 considers frequent
  - A_1 has to succeed in solving $\mathrm{re}(X)$

Summary

- P[A_0 fails] < **$1/x^3$**
- Possible A_1-failures:
  - P[unlucky] < **1/x**
  - P[A_0 fails] < **$1/x^3$**
  - P[A_0 outputs some infrequent label] < **#labels · #edges · $1/x$**
- P[A_1 fails] < **constant · $1/x$**

Randomized round elimination in cycles

- **A_0:** local failure probability < **$1/x^3$**
- **A_1:** local failure probability < **constant · $1/x$**
- Failure probability increases polynomially
- We can repeat this many times before A_k becomes useless

What works very often

- Do round elimination in deterministic PN model
  - gain intuition on how the problem behaves
- Then switch to randomized PN model
  - proper analysis of failure probabilities
- Results for deterministic & randomized LOCAL follow directly

Case study: **Coloring directed cycles**

A a
B b
C c

A bc
B ac
C ab

F G
E H
D I
C J
B K
A L

ACE GIK
BCF HIL
DEF JKL

A a
B b
C c
D d
E e
F f

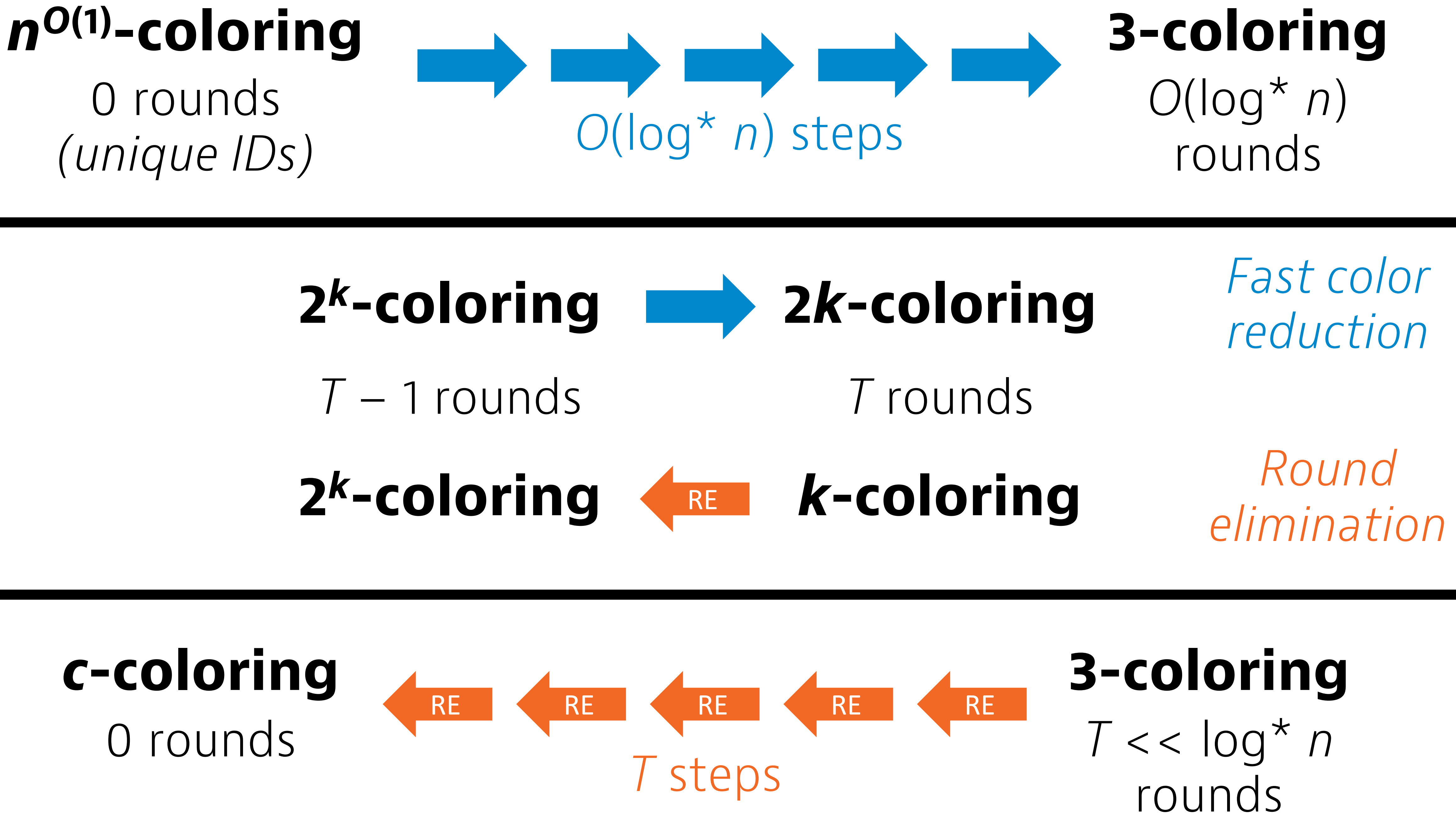
ACE fdb
BCF eda
DEF cba

A a
B b
C c
D d
E e
F f

A bdf
B ade
C abdef
D abc
E abcdf
F abcde

A a
B b
C c
D d
E e
F f

A bcdef
B acdef
C abdef
D abcef
E abcdf
F abcde


$n^{O(1)}$-coloring
0 rounds
(unique IDs)
$O(\log^* n)$ steps
3-coloring
$O(\log^* n)$
rounds
2^k-coloring
$2k$-coloring
Fast color reduction
$T - 1$ rounds
T rounds
2^k-coloring
RE
k-coloring
Round elimination
c-coloring
0 rounds
RE
RE
RE
RE
RE
T steps
3-coloring
$T << \log^* n$
rounds

Sinkless orientation

- **Deterministic PN:**
  - not possible in $o(\log n)$ rounds (last week)
  - possible in $O(\log n)$ rounds (last week)
- **Randomized PN:**
  - not possible in $o(\log \log n)$ rounds (exercise)
  - possible in $O(\log \log n)$ rounds (not easy)
- **Deterministic LOCAL?**