Labelling grids locally

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- Philosophically, the conjecture $P \subset PSPACE$ proposes that space is more complex than time.
- Other examples include recent work on the black hole firewall paradox.

Aims

- Our broad goal is to take a similar look at reality through the lens of distributed computing.
- In distributed computing, our limited resources are information about the input and communication.
- We want to understand the distributed computational capabilities of reality.

Grids

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- Grids are a reasonable place to start and are nice to reason about.
- Typical lower bound constructions are good expanders, grids are more realistic.
- Grids haven't been well studied from our point of view.

Our model

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- A collection of nodes communicate in synchronous rounds and try to solve a problem.
- Each node sits at a vertex of a graph G and in each round communicates with its neighbours.
- The nodes may have some input (e.g. a unique identifier) but do not know G initially.
- Local computation is free; the complexity measure is the number of rounds.

Our model

- Input graph is an $n \times n$ grid with a consistent orientation unless otherwise specified.
- Nodes have unique identifiers and must output a label according to some set of rules.
- We restrict ourselves to labellings that are *locally checkable*: validity in the constant radius around each node implies global validity; e.g., vertex colouring, edge colouring, maximal independent set (MIS)

Warm up: one dimension

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Theorem

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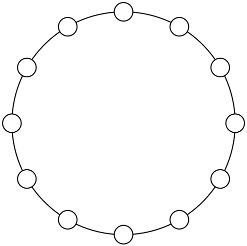
Idea: for suitable constant k, we find a MIS in G^k in $O(log^* n)$ time and try to fix a particular labelling for each possible gap.



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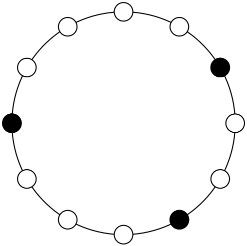

Figure: An MIS in G^3

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Theorem

On an oriented cycle, every solvable locally checkable labelling problem has asymptotic complexity $O(1)$, $\Theta(log^* n)$ or $\Theta(n)$ time.

Definition

A vertex v in a directed graph is *flexible* if there exists a constant k such that for all $k' > k$ there is a circuit of length k that begins and ends at v.

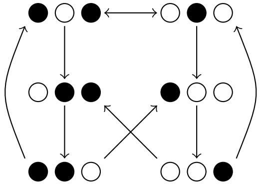

Figure: Locally optimal cut

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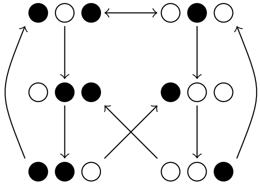
Proof (Sketch).

If the *neighbourhood graph* of Π has a flexible label we can use the MIS. If there is no such label, then for some b nodes at distance $0 mod b$ must have the same label.

□



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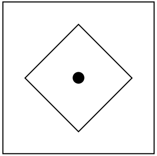

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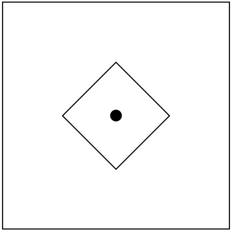

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Given algorithm A that runs in $o(r)$ on $r \times r$ grids we want an algorithm that runs in $O(\log^* n)$ for $n \times n$ grid G.

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Proof (Sketch).

Given algorithm A that runs in $o(r)$ on $r \times r$ grids we want an algorithm that runs in $O(\log^* n)$ for $n \times n$ grid G.

- Pick suitable constants r, k.
- Find an MIS in G^k, and use it to pick *locally unique identifiers* for $r \times r$ neighbourhoods.
- Simulate A.

□

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Question

Can we characterise the classes of LCLs having asymptotic complexity $O(1), \Theta(log^* n), \Theta(n)$?

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Not in general - whether a given LCL has complexity $\Theta(log^* n)$ or $\Theta(n)$ on grids is undecidable.

Synthesis

If we know that a problem has $\Theta(log^* n)$ complexity on grids (or we make a lucky guess) we can find such an algorithm.

- For each r, k we enumerate all radius r neighbourhoods that represent possible fragments of an MIS in G^k.
- Construct the neighbourhood graph - an algorithm is a labelling of the neighbourhood graph.
- Use SAT solvers to find a labelling.

Results: vertex colouring

A 2-colouring may not exist in a grid, and is inherently a global problem. A 5-colouring of a grid is a $\Delta + 1$-colouring

- 2-colouring: $\Theta(n)$
- 3-colouring: ??
- 4-colouring: ??
- 5-colouring: $\Theta(log^* n)$

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$d=1$	0	1	2
0	$O(1)$	$O(1)$	$O(1)$
1	$O(1)$	$\Theta(log^* n)$	$\Theta(log^* n)$
2	$O(1)$	$\Theta(log^* n)$	$\Theta(n)$

Figure: Asymptotic complexity of a, b-labellings for oriented cycles

$d=2$	0	1	2	3	4
0	$O(1)$	$O(1)$	$O(1)$	$O(1)$	$O(1)$
1	$O(1)$	$\Theta(log^* n)$	$\Theta(log^* n)$	$\Theta(log^* n)$	$\Theta(log^* n)$
2	$O(1)$	$\Theta(log^* n)$	$\Theta(log^* n)$	$\Theta(log^* n)$	$\Theta(n)$
3	$O(1)$	$\Theta(log^* n)$	$\Theta(log^* n)$	$\Theta(n)$	$\Theta(n)$
4	$O(1)$	$\Theta(log^* n)$	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$

Figure: Asymptotic complexity of a, b-labellings for oriented 2d grids

Summary

- For directed grids all LCL problems have a $O(log^* n)$ upper bound or $\Omega(n)$ lower bound.
- 4-colouring is $\Theta(log^* n)$, 3-colouring is $\Theta(n)$.
- $O(log * n)$ algorithms can be synthesised.

Questions

- Interpretation: what's the moral of the story?
- Connections to physics and real world systems.
- What next?