Deterministic distributed algorithms: using covering graphs for good and evil

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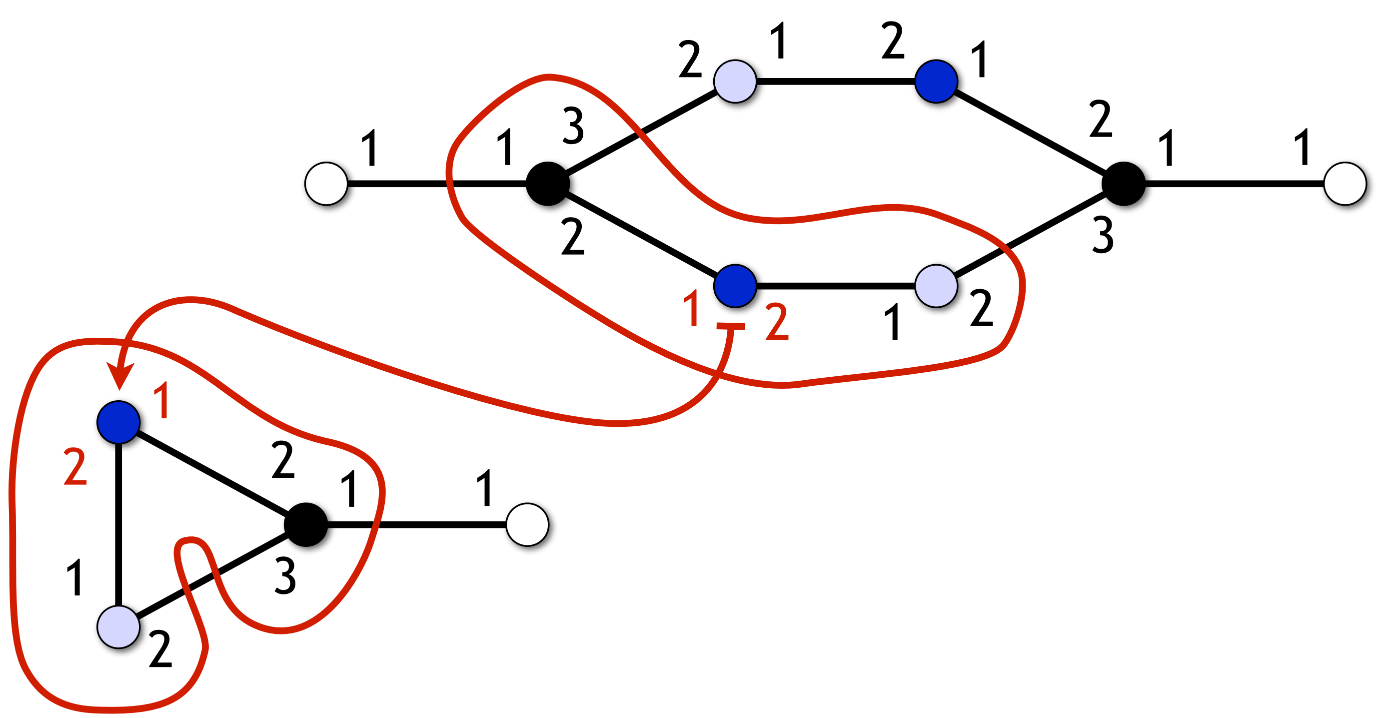
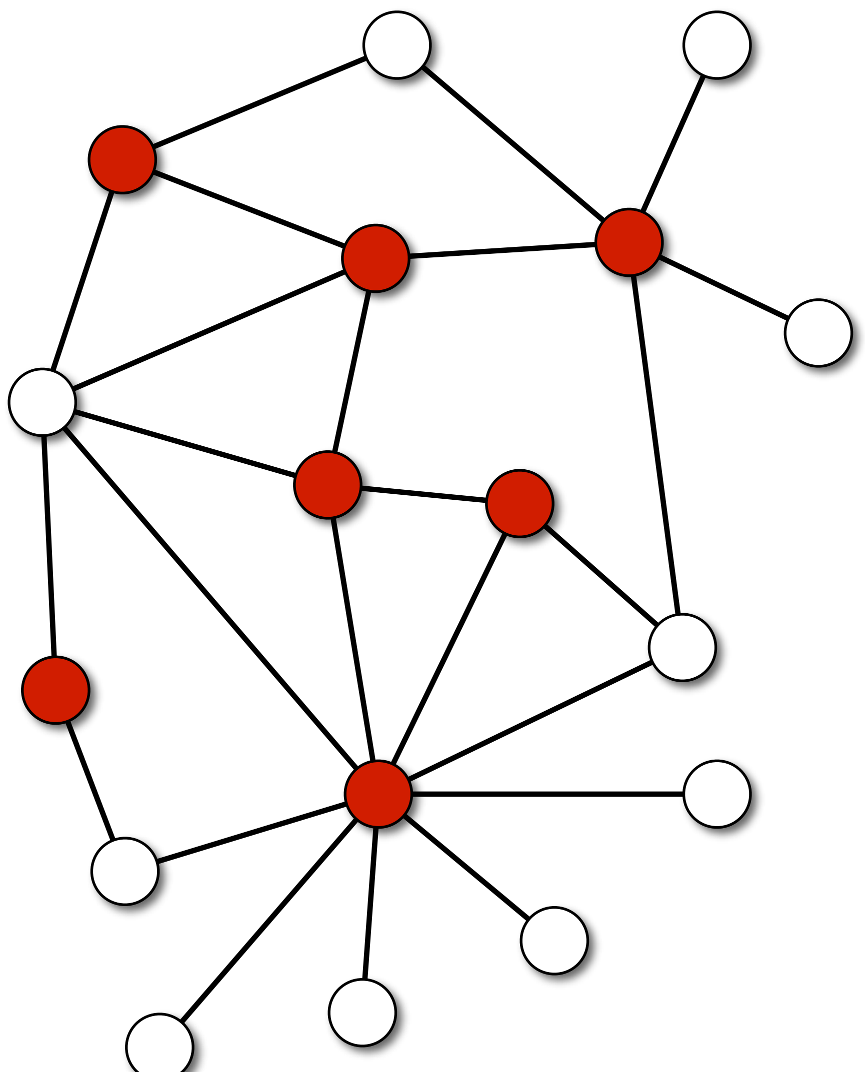


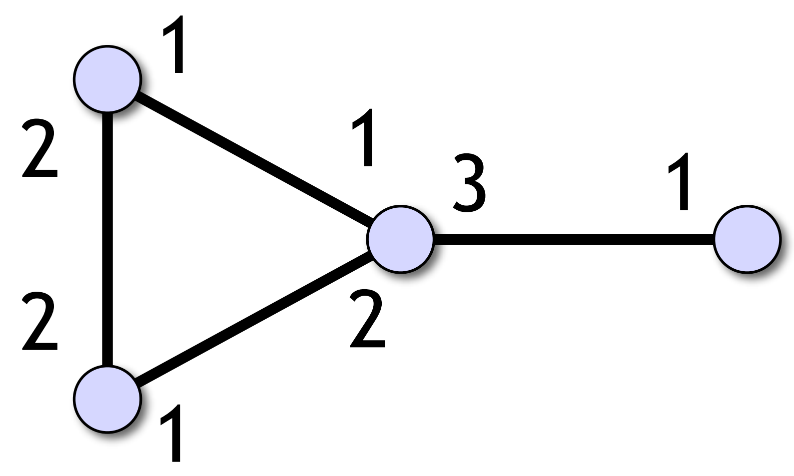
Running example: Vertex cover problem



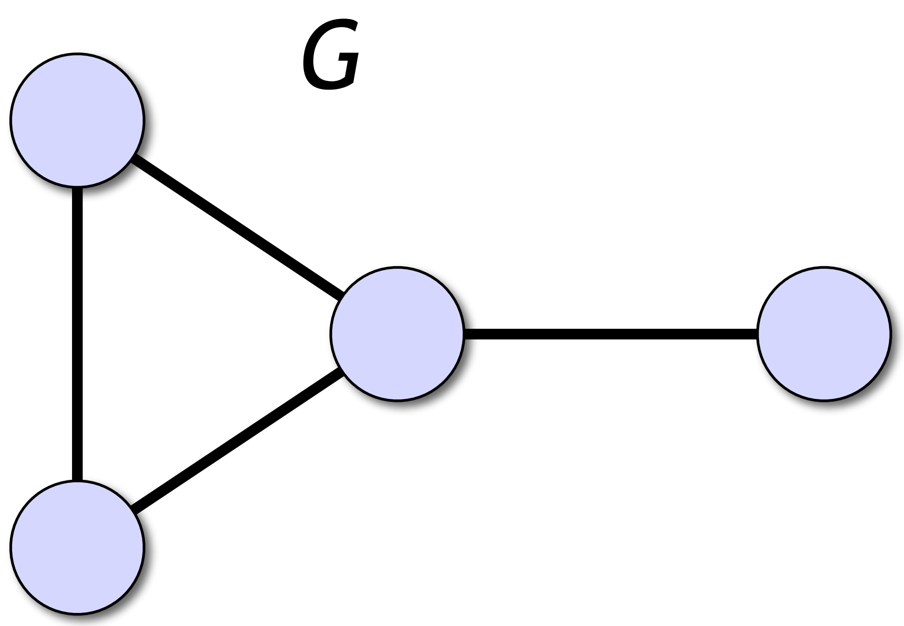
- Vertex cover *C*:
  - "covers" all edges of the graph
  - each edge has at least one endpoint in *C*

Part I: Port-numbering model

- Synchronous deterministic distributed algorithms in the port-numbering model

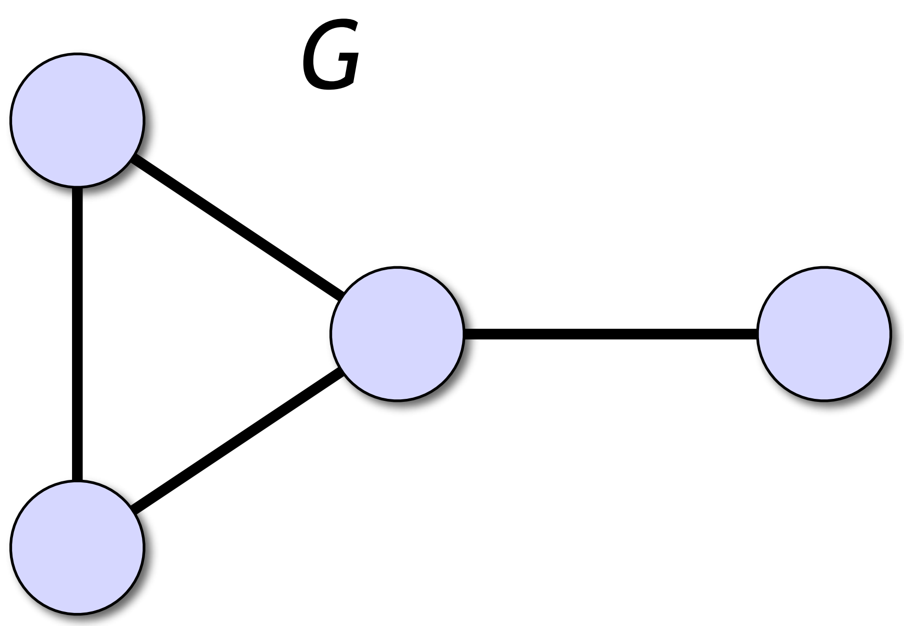


Distributed algorithms



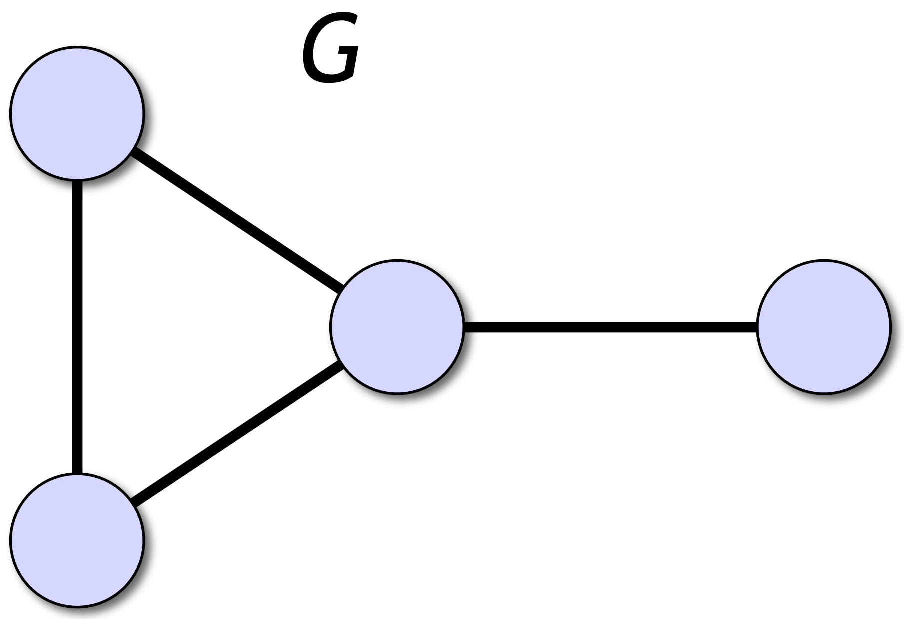
- Communication graph *G*
- Node = computer
  - e.g., Turing machine, finite state machine
- Edge = communication link
  - computers can exchange messages

Distributed algorithms



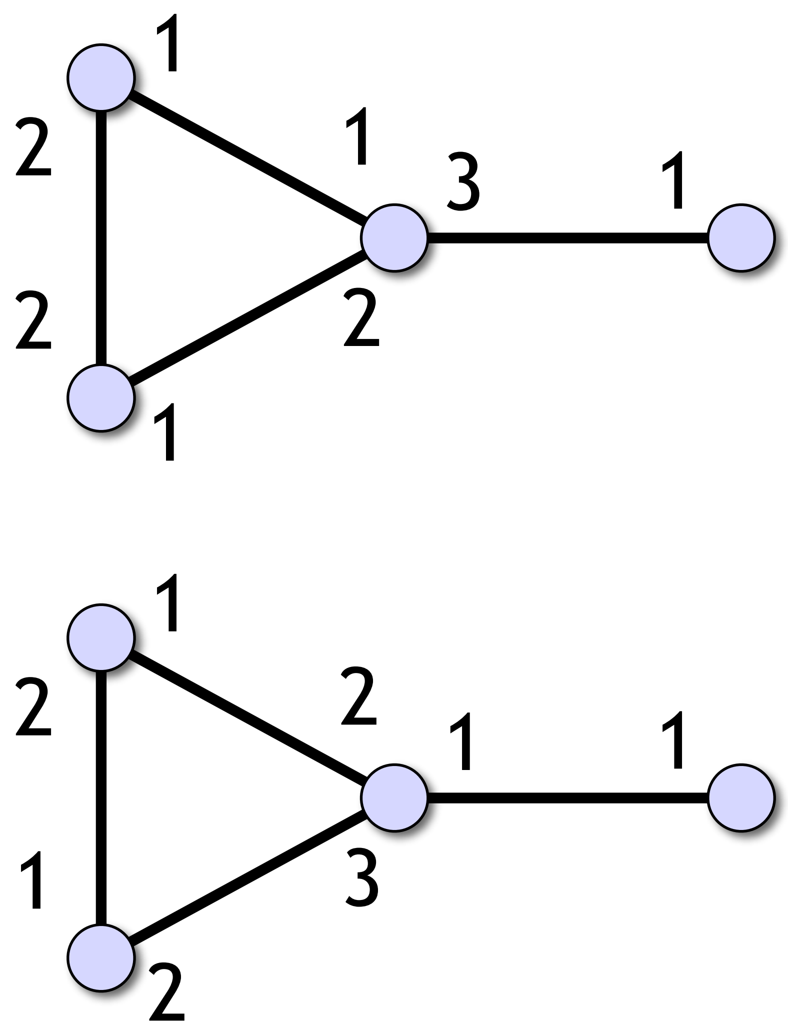
- All nodes are identical, run the **same algorithm**
- ***We*** can choose the algorithm
- An ***adversary*** chooses the structure of *G*
- Our algorithm must produce a correct output in any graph *G*

Distributed algorithms



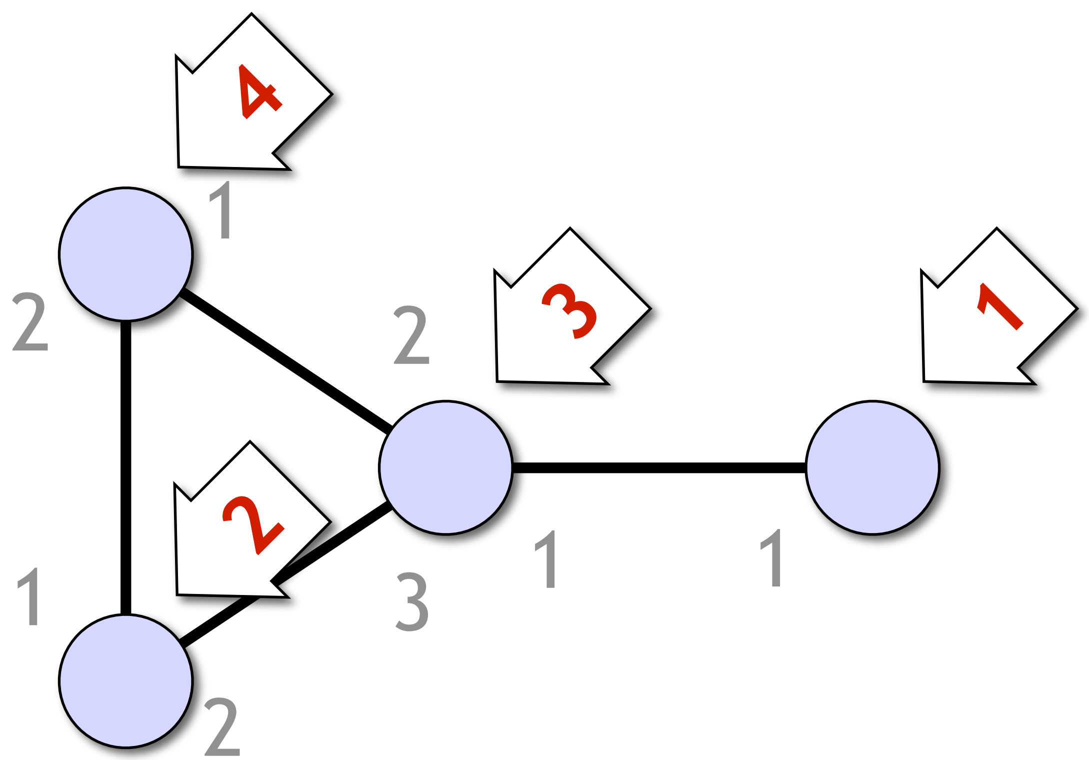
- Usually, computational problems are related to the structure of the communication graph *G*
  - example: find a vertex cover for *G*
  - the same graph is both the input and the system that tries to solve the problem...

Port-numbering model



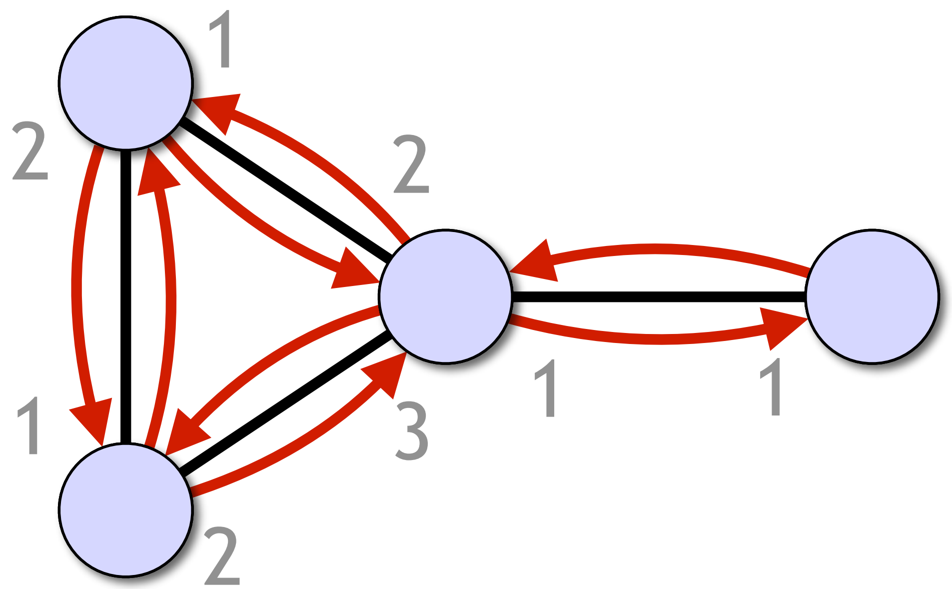
- A node of degree *d* can refer to its neighbours by integers 1, 2, ..., *d*
- Port-numbering chosen by adversary

Synchronous distributed algorithms



1. Each node reads its own **local input**
   - Depends on the problem, for example:
     - node weight
     - weights of incident edges
   - May be empty

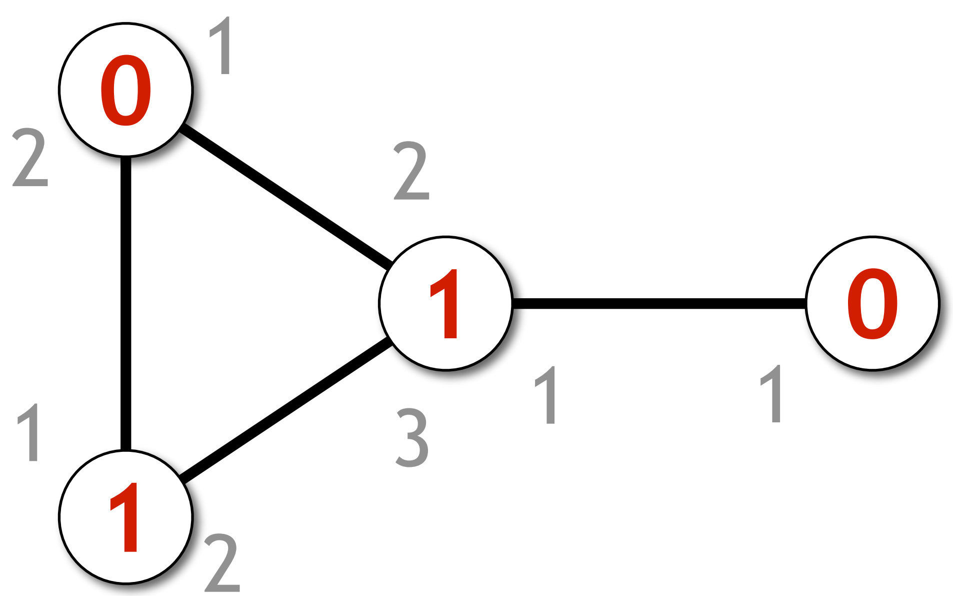
Synchronous distributed algorithms



1. Each node reads its own **local input**
2. Repeat synchronous **communication rounds**

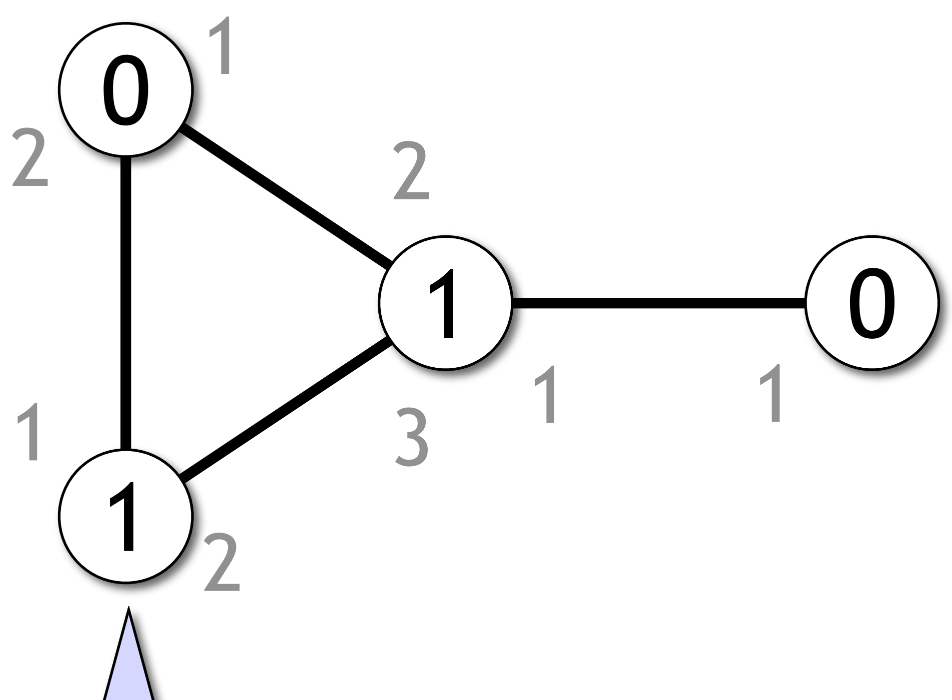
   …

Synchronous distributed algorithms



1. Each node reads its own **local input**
2. Repeat synchronous **communication rounds** until all nodes have announced their **local outputs**
   - Solution of the problem

Synchronous distributed algorithms

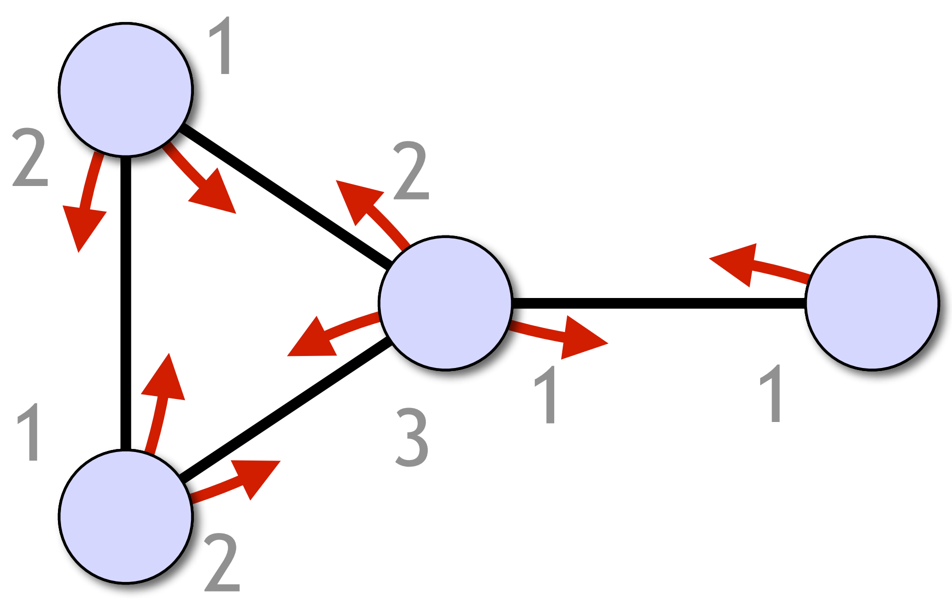


1. Each node reads its own **local input**
2. Repeat synchronous **communication rounds** until all nodes have announced their **local outputs**

Example: Find a vertex cover C

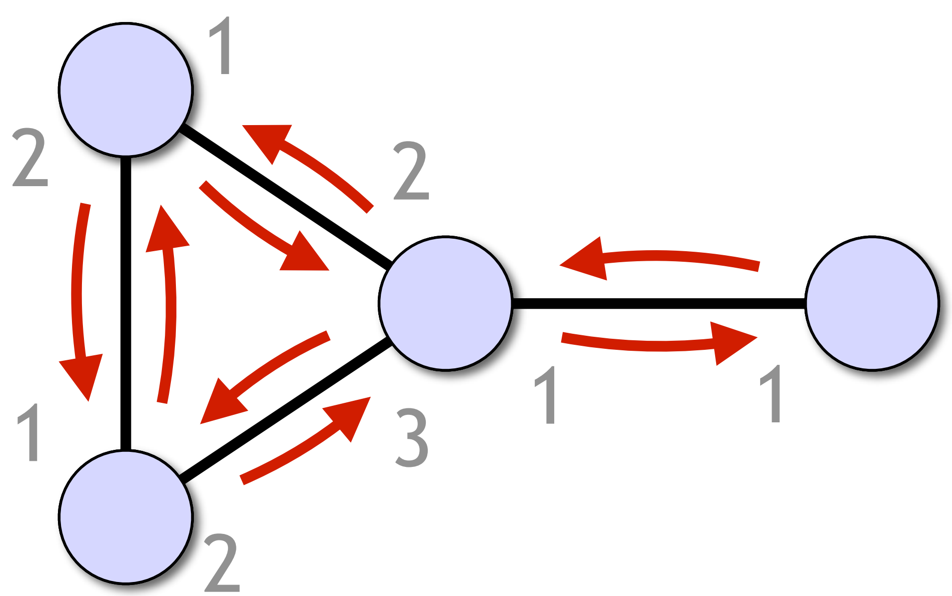
Local output of a node v indicates whether $v \in C$

Synchronous distributed algorithms



- Communication round: each node
    1. sends a message to each port

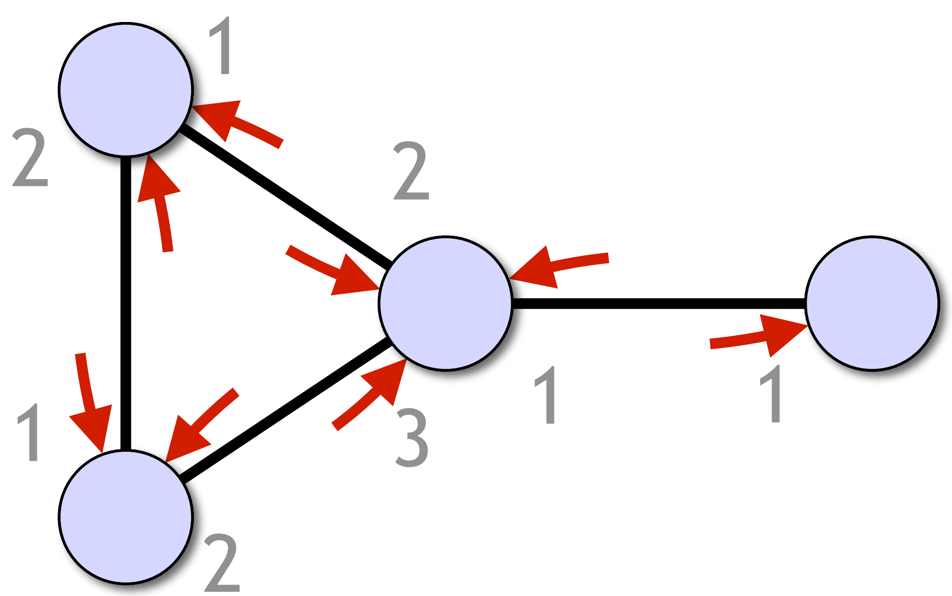
Synchronous distributed algorithms



- Communication round: each node
    1. sends a message to each port

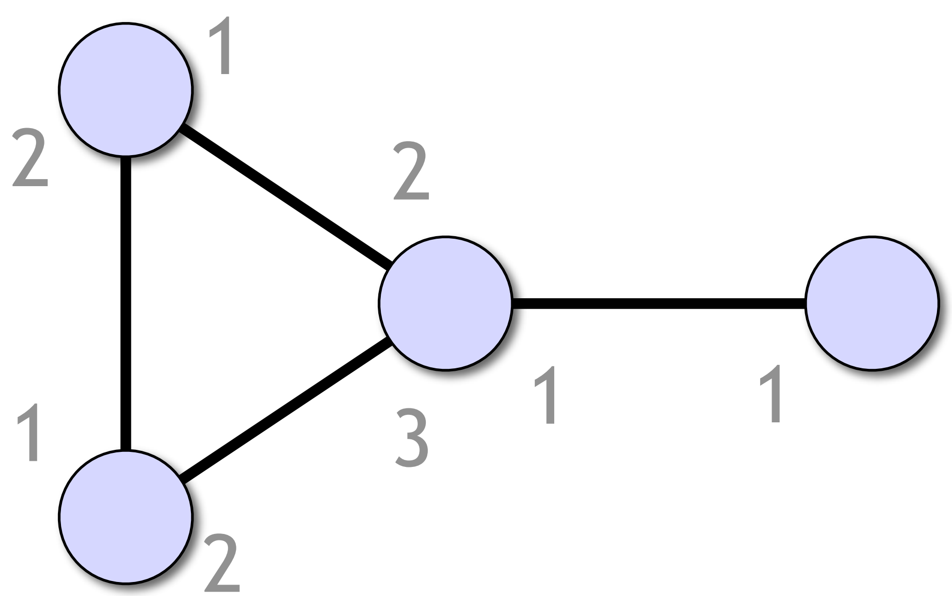
    (message propagation...)

Synchronous distributed algorithms



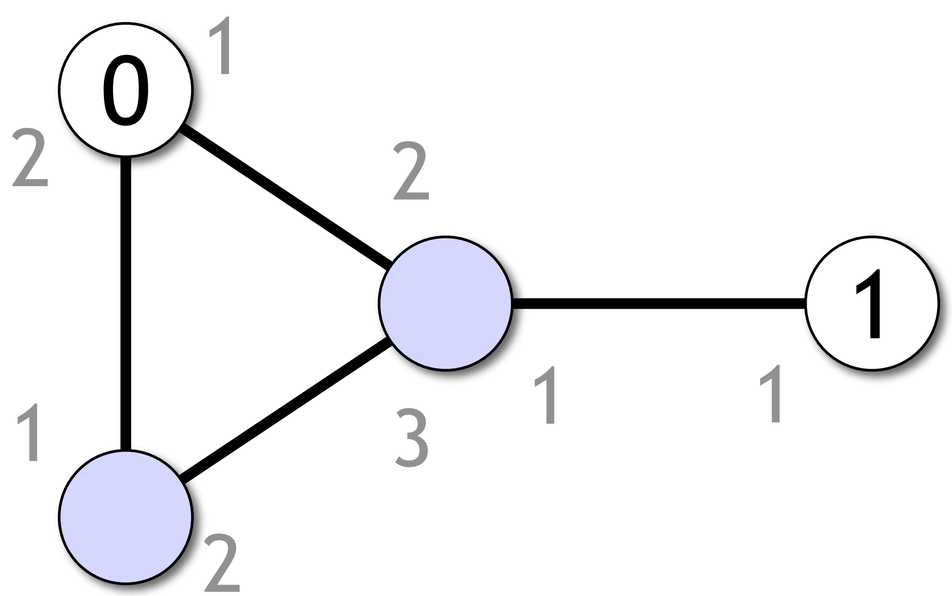
- Communication round: each node
  1. sends a message to each port
  2. receives a message from each port

Synchronous distributed algorithms



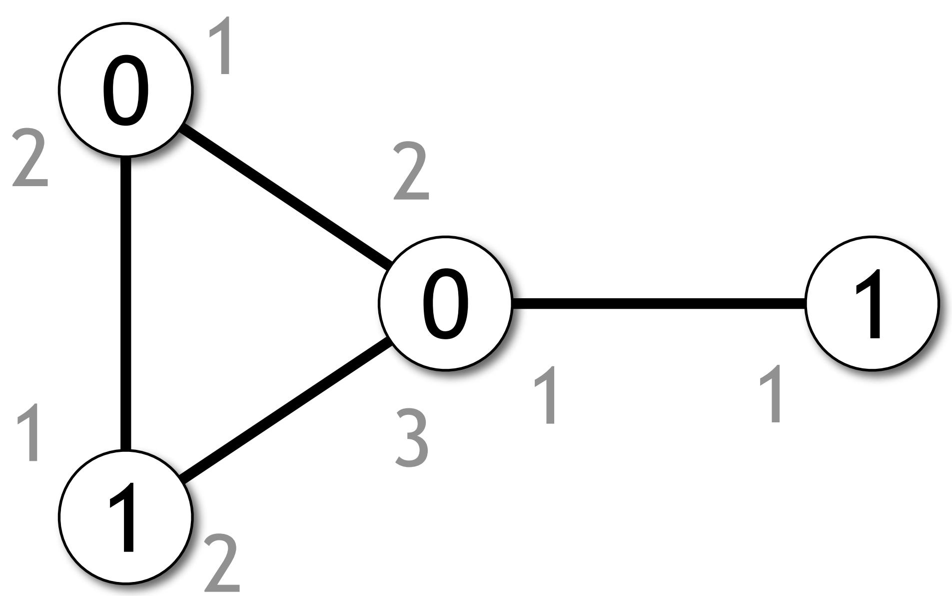
- Communication round: each node
  1. sends a message to each port
  2. receives a message from each port
  3. updates its own state

Synchronous distributed algorithms



- Communication round: each node
  1. sends a message to each port
  2. receives a message from each port
  3. updates its own state
  4. possibly stops and announces its output

Synchronous distributed algorithms

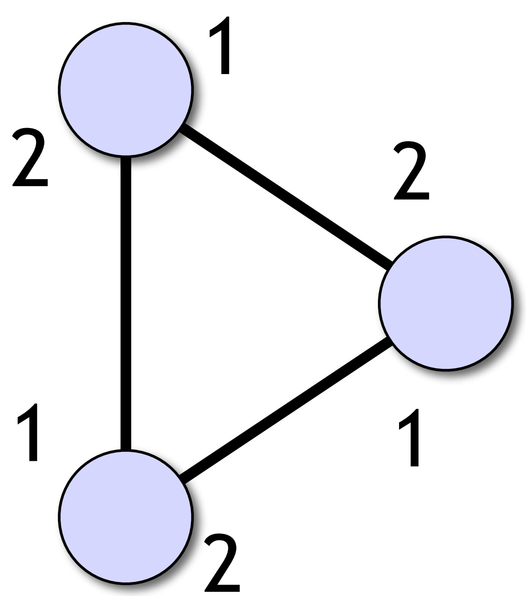


- Communication rounds are repeated until all nodes have stopped and announced their outputs
- Running time = **number of rounds**
- Worst-case analysis

Part II: Computability in port-numbering model

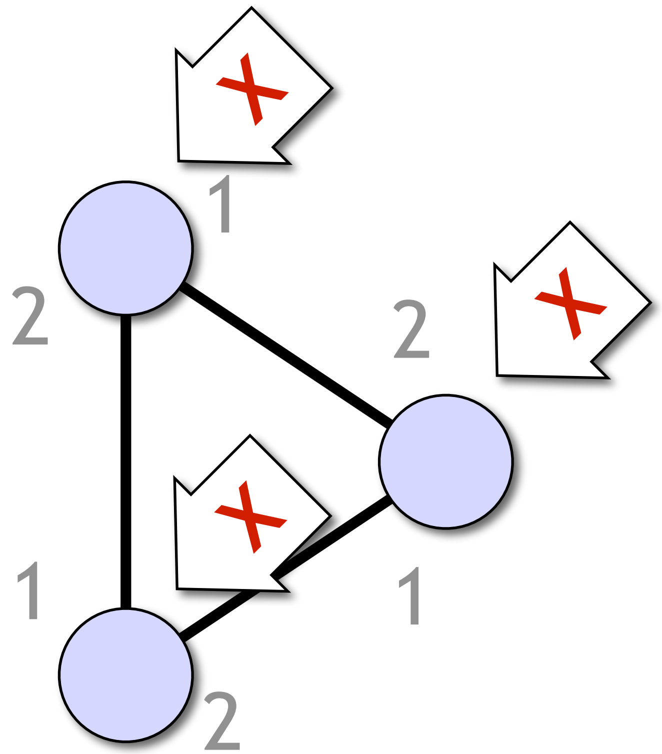
- Impossibility of symmetry breaking
- Covering maps and covering graphs: tools for proving more impossibility results

Symmetry can't be broken



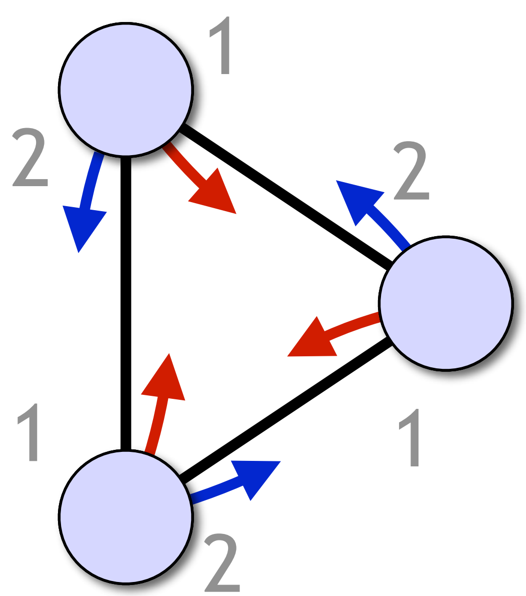
- Input may be symmetric
  - symmetric graph
  - symmetric port numbering
  - identical local inputs

Symmetry can’t be broken



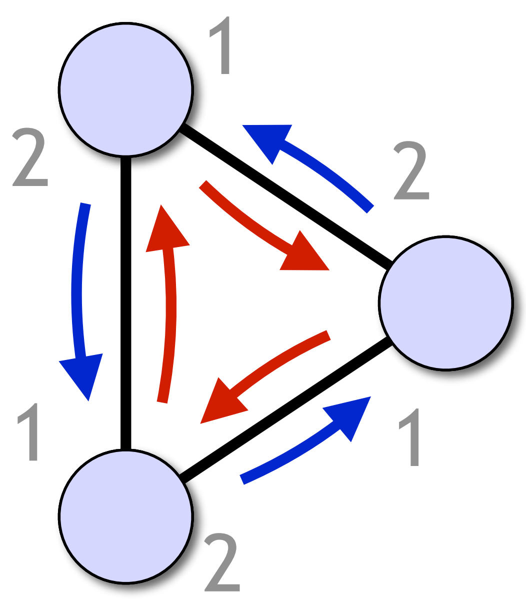
- Same input
- Same algorithm
- Same initial state

Symmetry can’t be broken



- Same current state
- Messages sent to port 1 are identical to each other
- Messages sent to port 2 are identical to each other

Symmetry can't be broken



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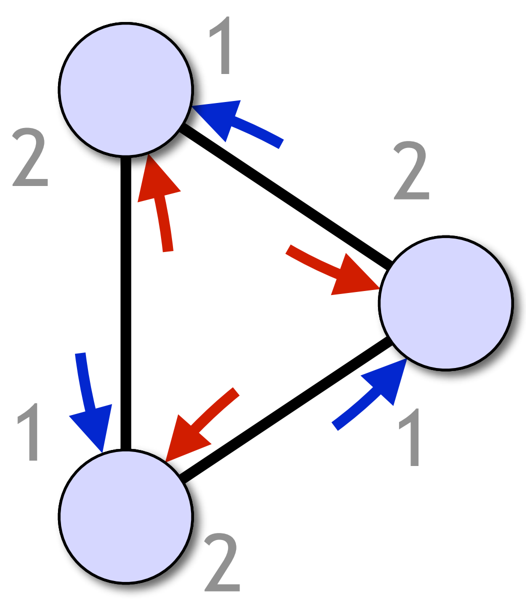
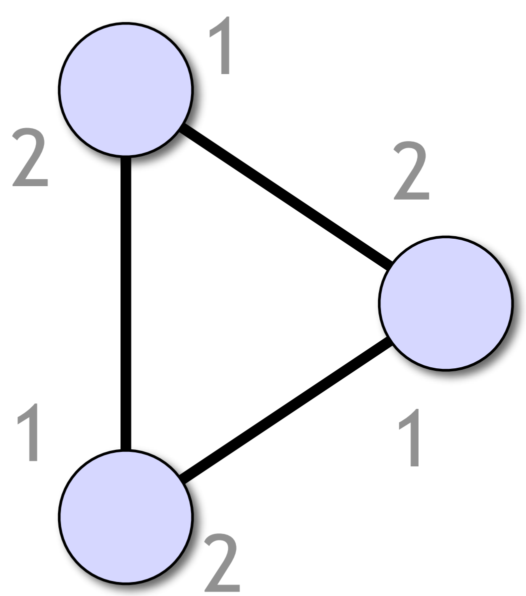


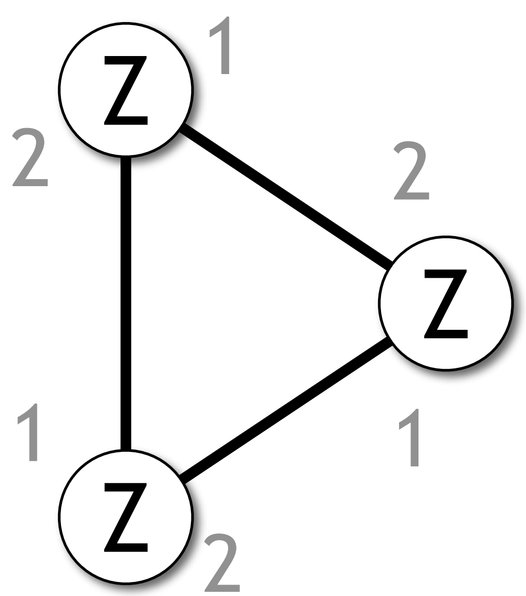
- Messages received from port 1 are identical to each other
- Messages received from port 2 are identical to each other

Symmetry can't be broken



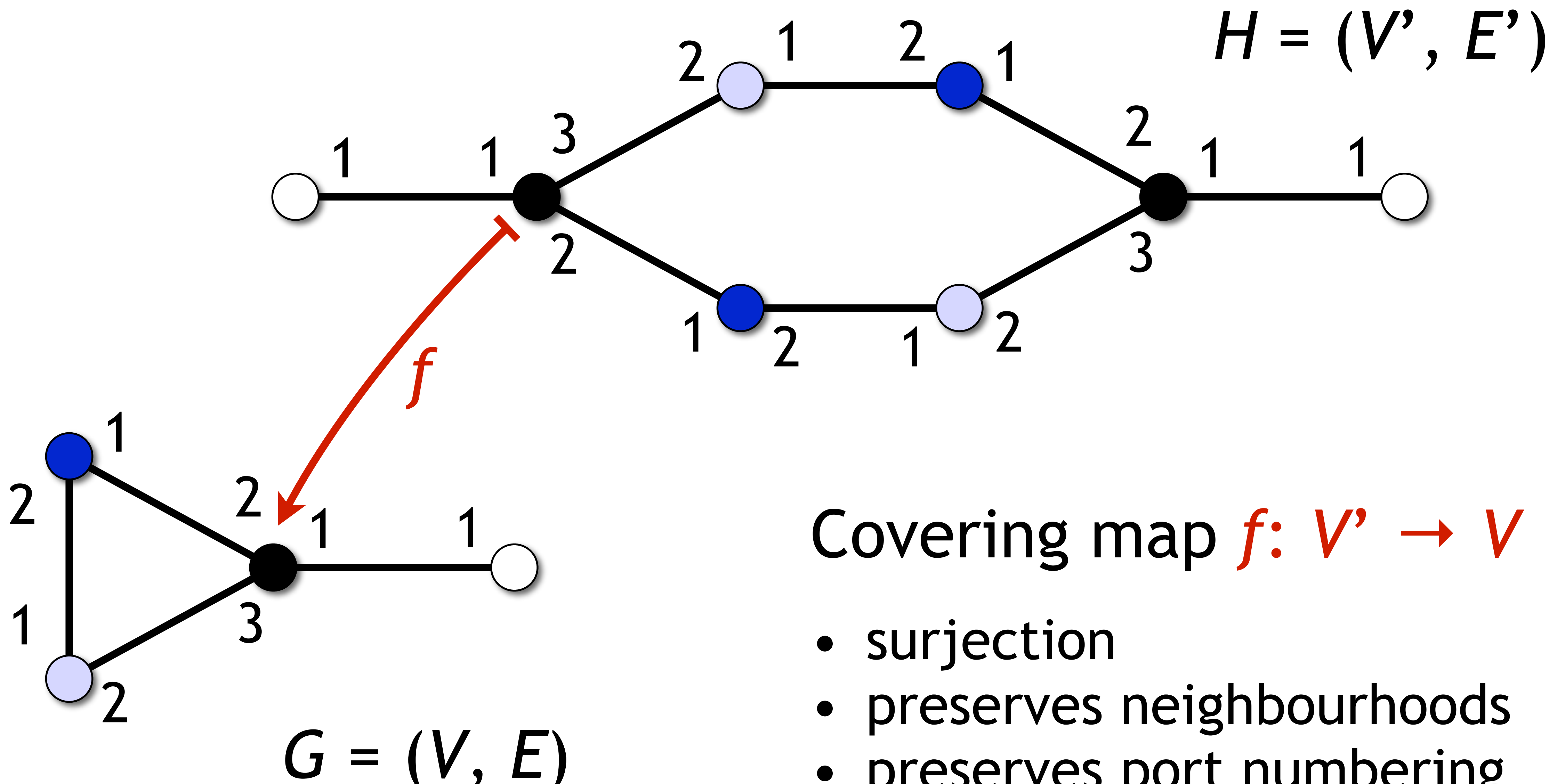
- Same old state
- Same set of received messages
- Same deterministic algorithm
- **Same new state**

Symmetry can’t be broken



- Same new state
- Either none of the nodes stops — or all of them stop and produce **identical outputs**
- Symmetry can’t be broken!
    - let’s generalise this...

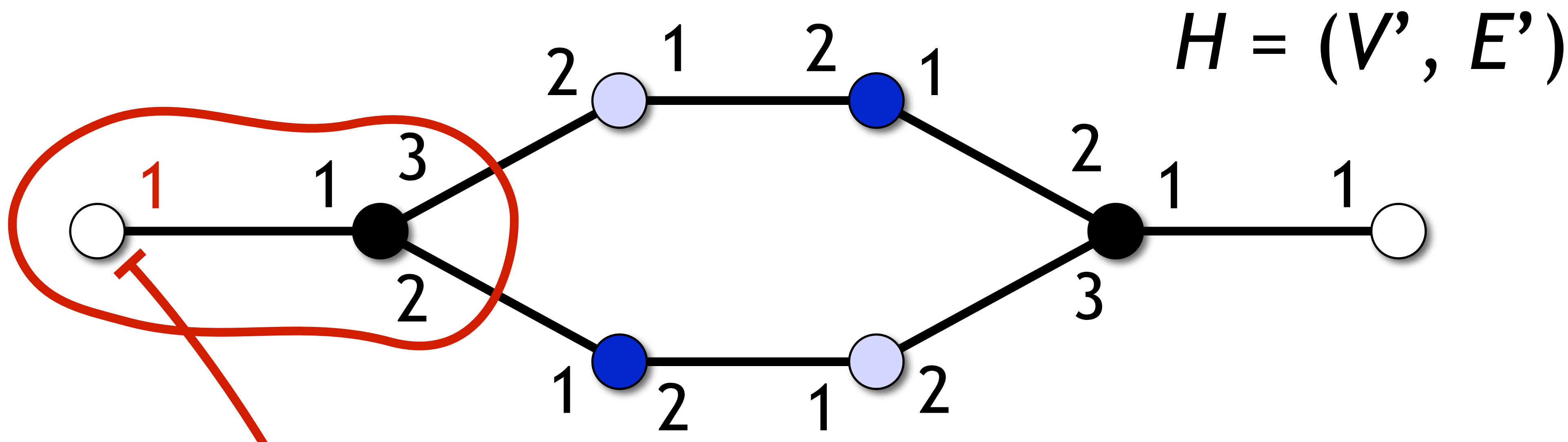
Covering maps

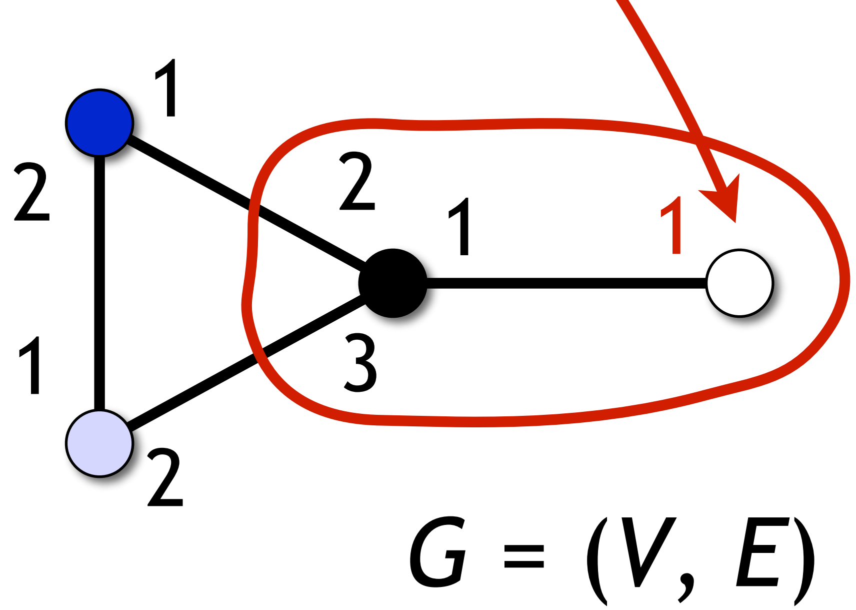


Covering map f: $V' \to V$

- surjection
- preserves neighbourhoods
- preserves port numbering

Covering maps





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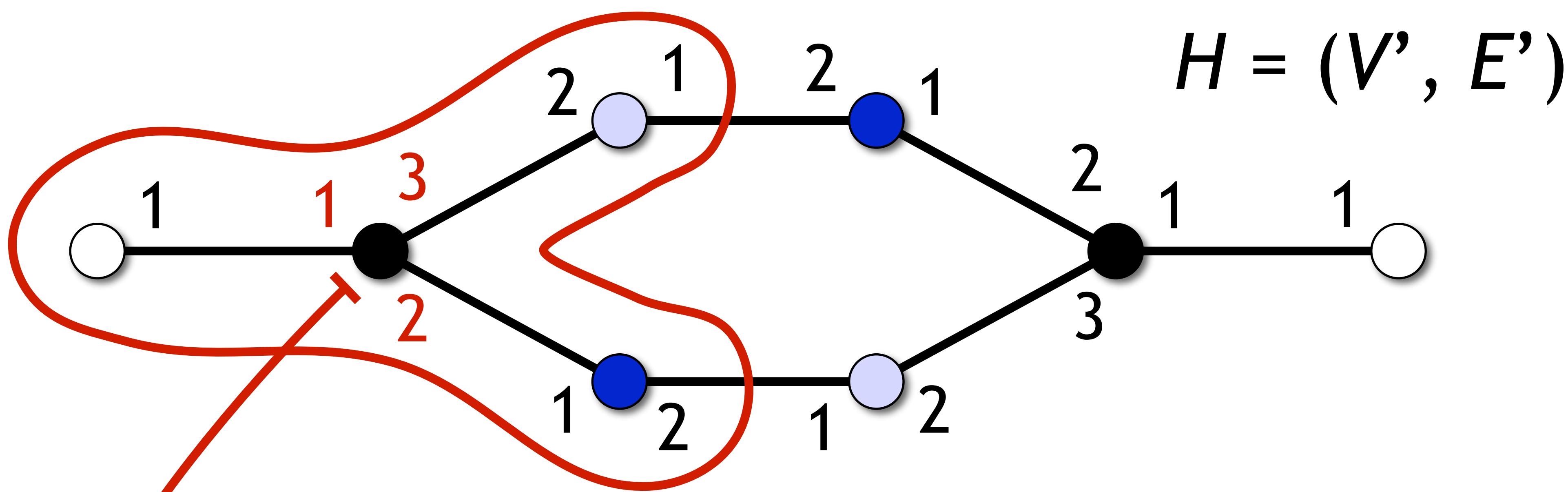
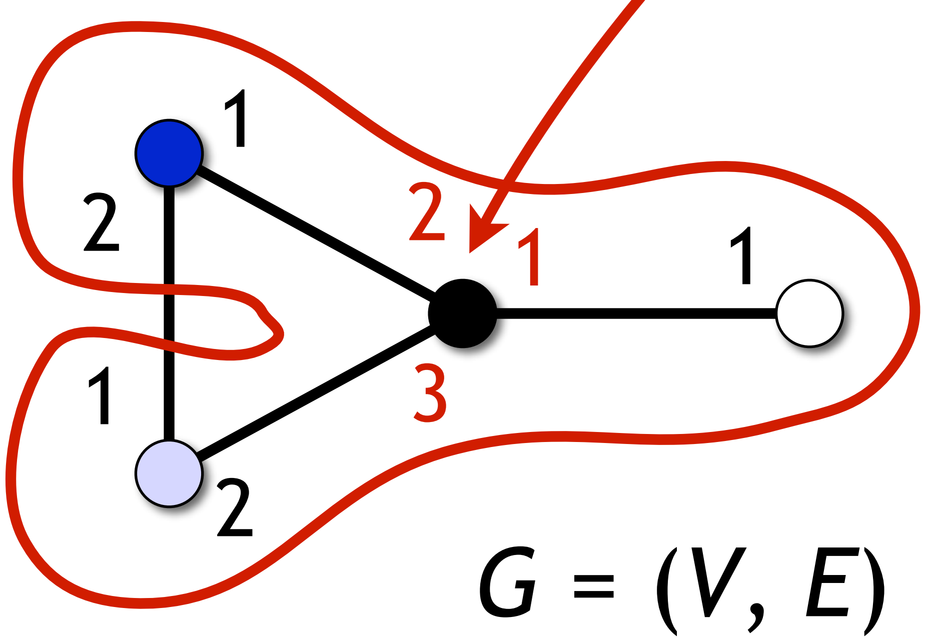
Covering maps





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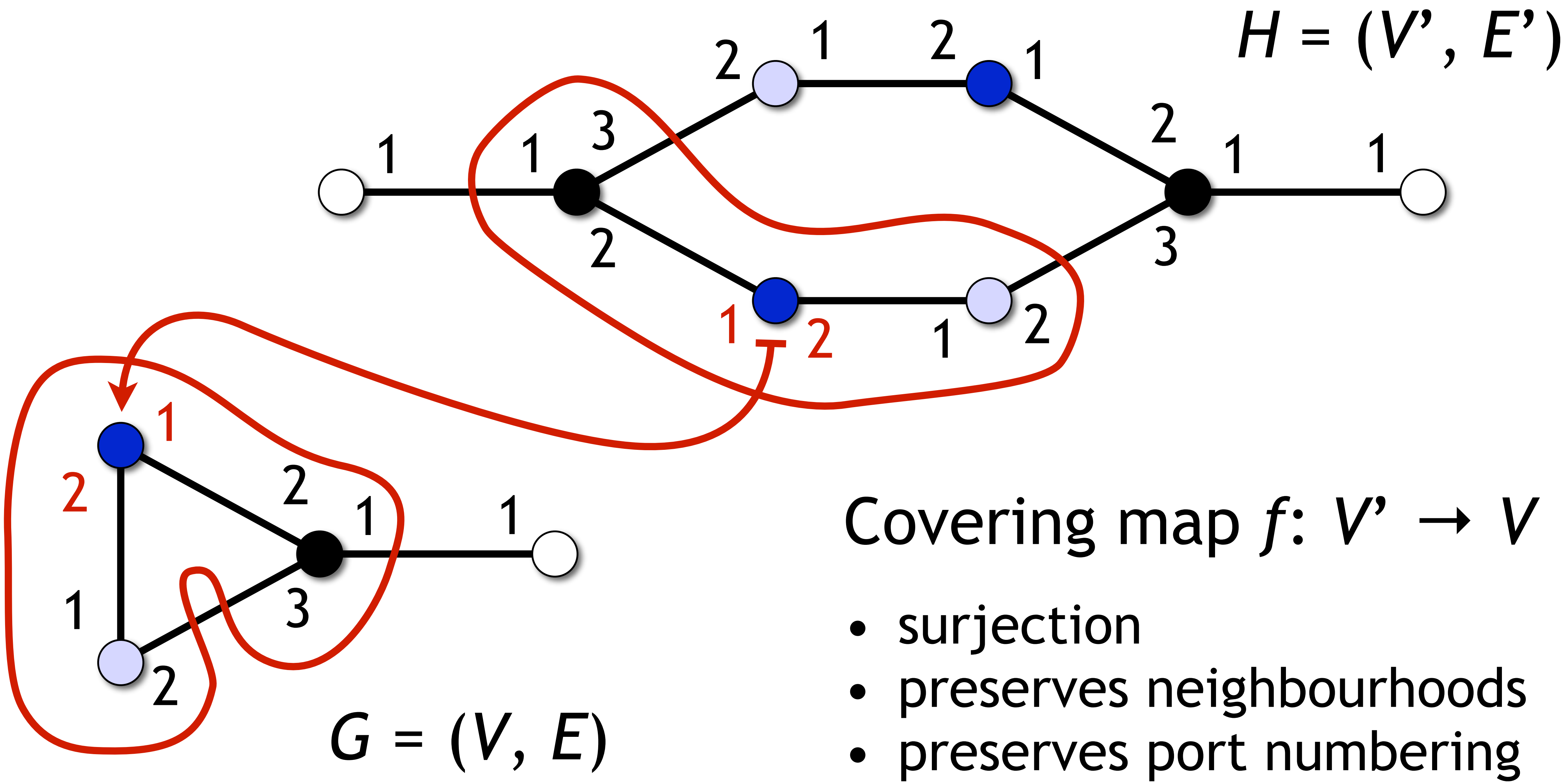
Covering maps



Covering map f: $V' \to V$

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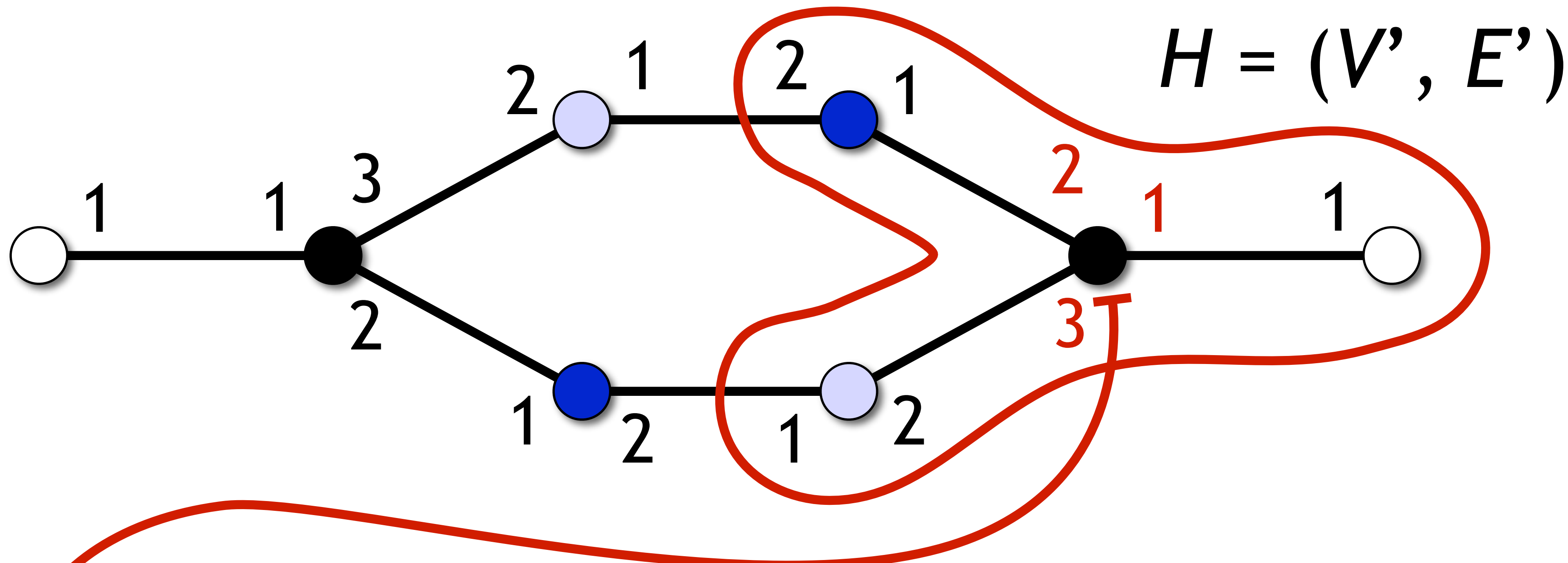
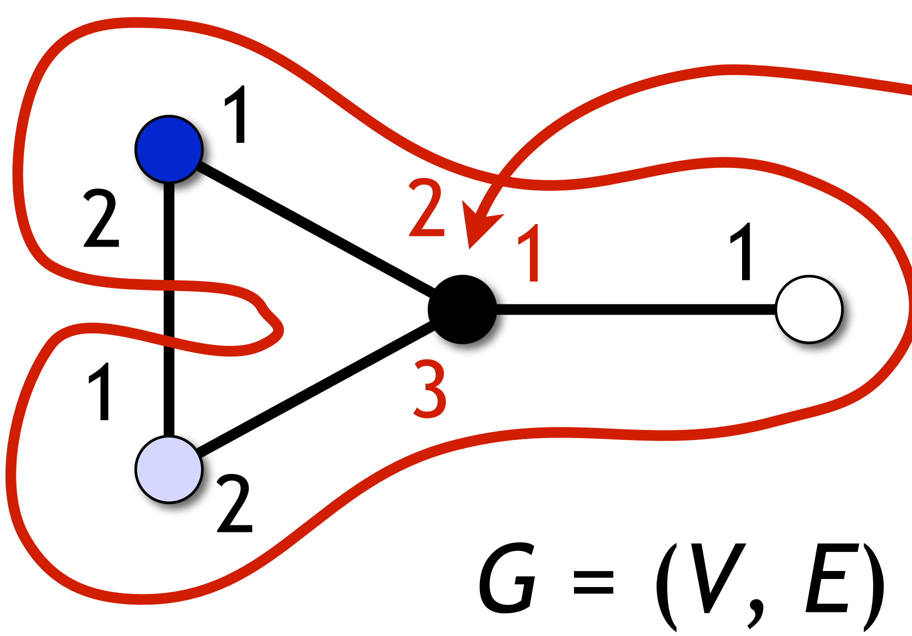
Covering maps

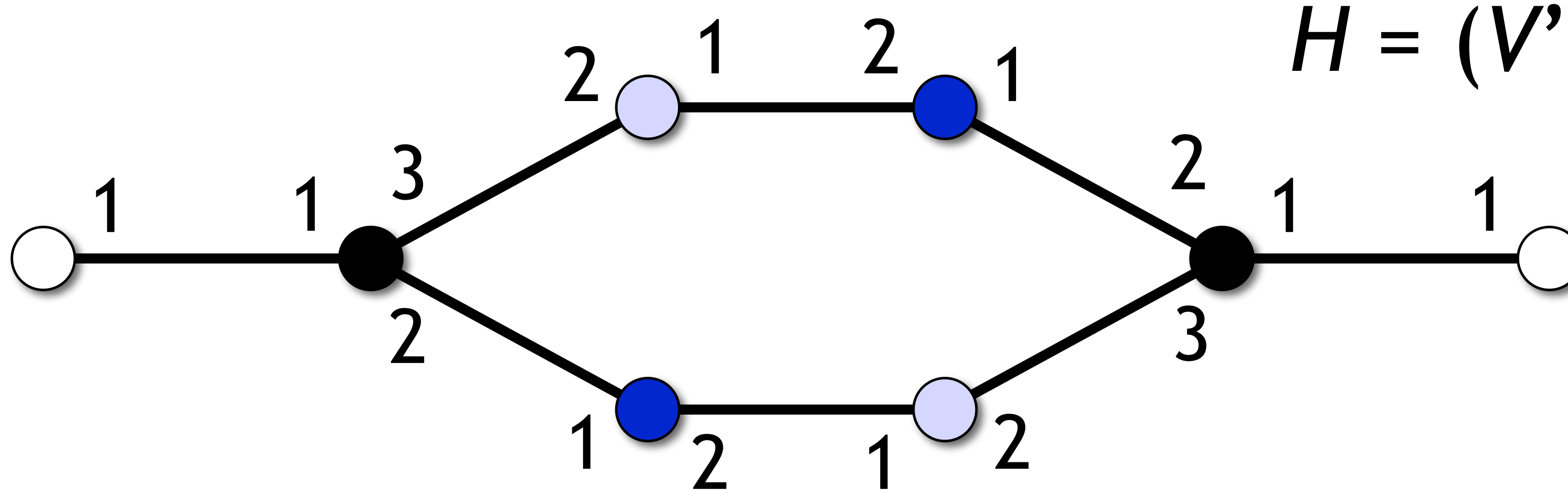




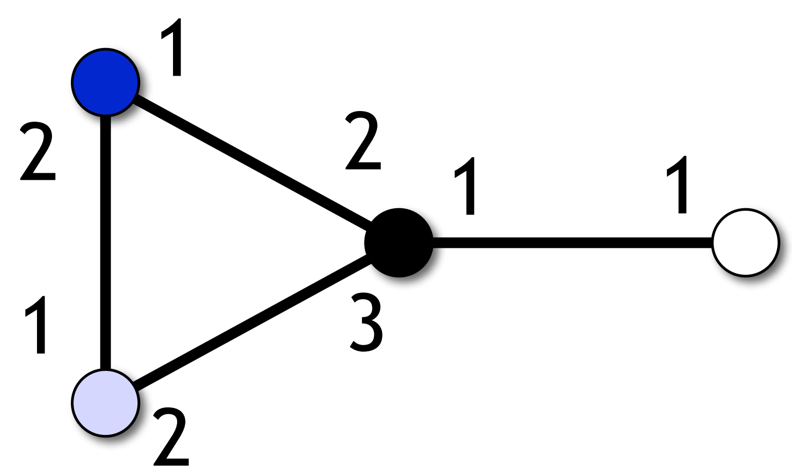
Covering map f: $V' \rightarrow V$

- surjection
- preserves neighbourhoods
- preserves port numbering

Covering maps and covering graphs



H = (*V*', *E*')



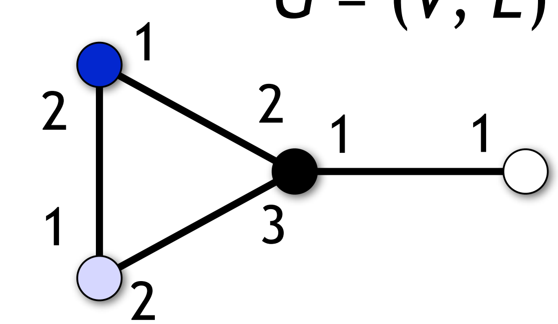
G = (*V*, *E*)

H is a **covering graph** of *G* if there is a covering map *f*: *V*' → *V*

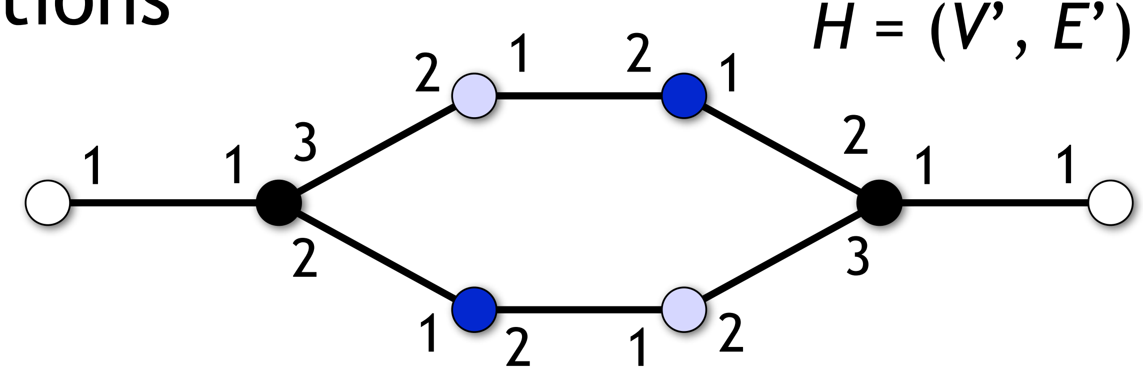
Covering maps and covering graphs

- Run the **same algorithm** in G and H
  - $v' \in V'$ and $f(v') \in V$ have the same input for all v'
- Then $v' \in V'$ and $f(v') \in V$:
  - have identical initial states
  - send and receive the same messages
  - have identical state transitions
  - produce **identical local outputs**!

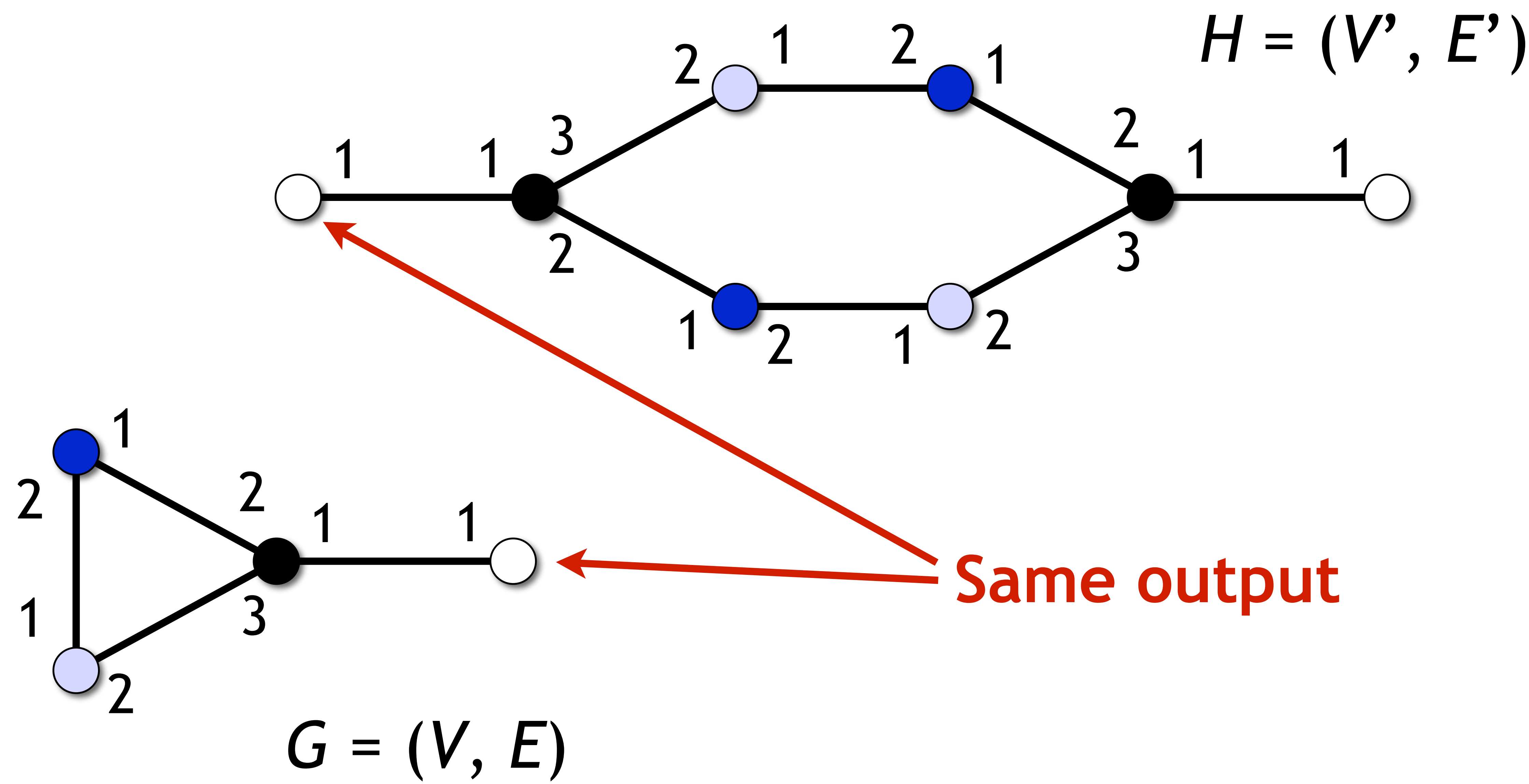
$G = (V, E)$



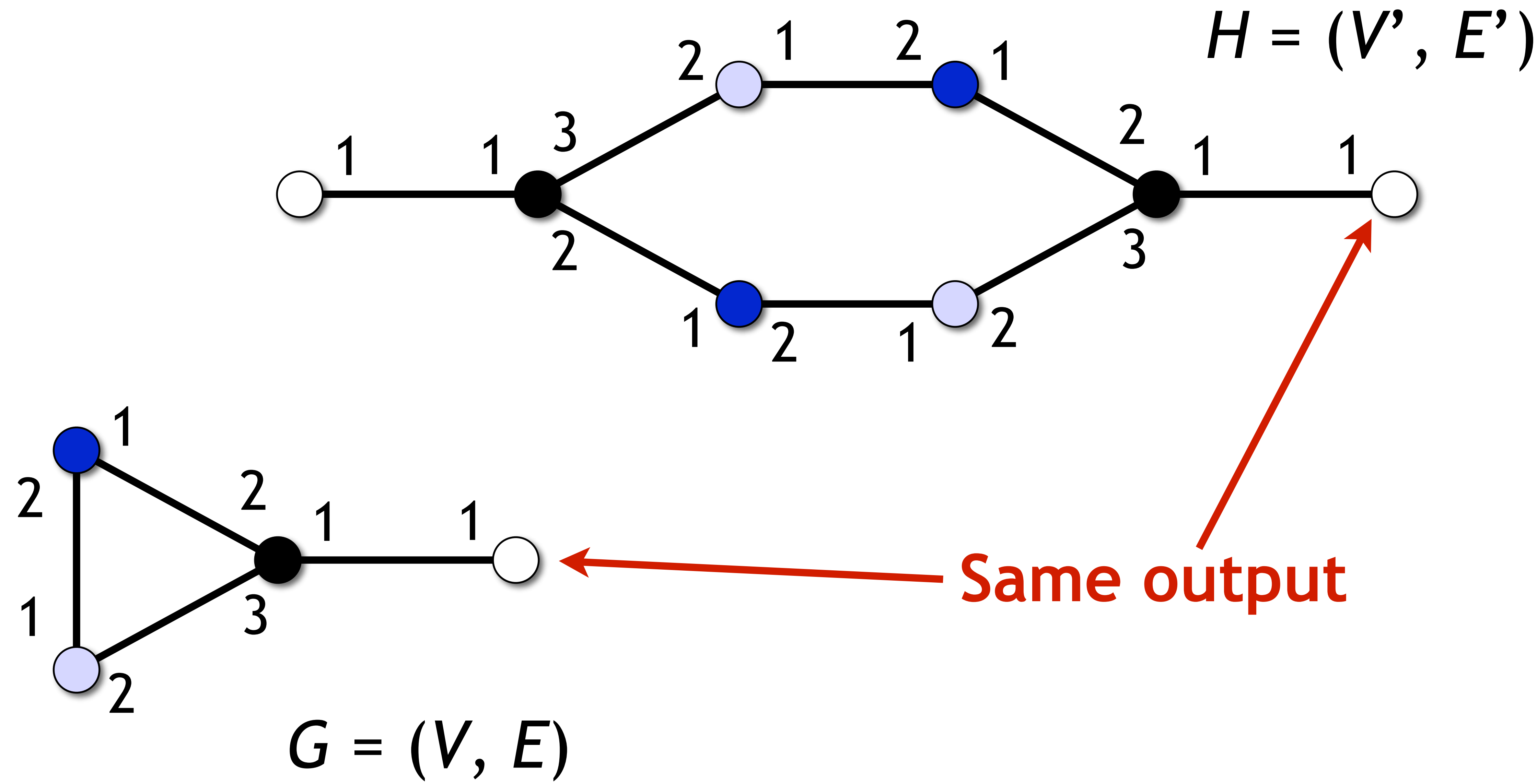
$H = (V', E')$



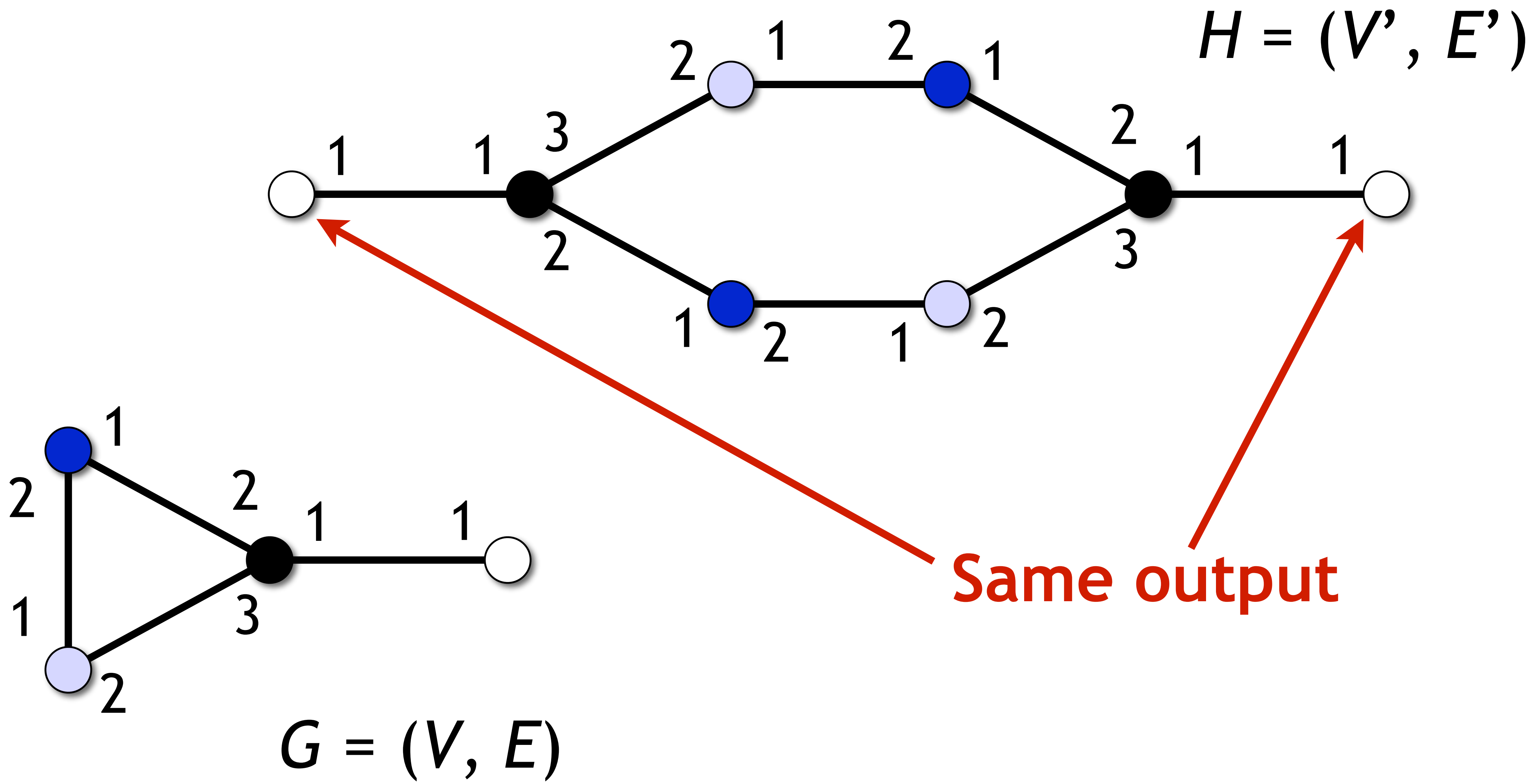
Covering maps and covering graphs



Covering maps and covering graphs

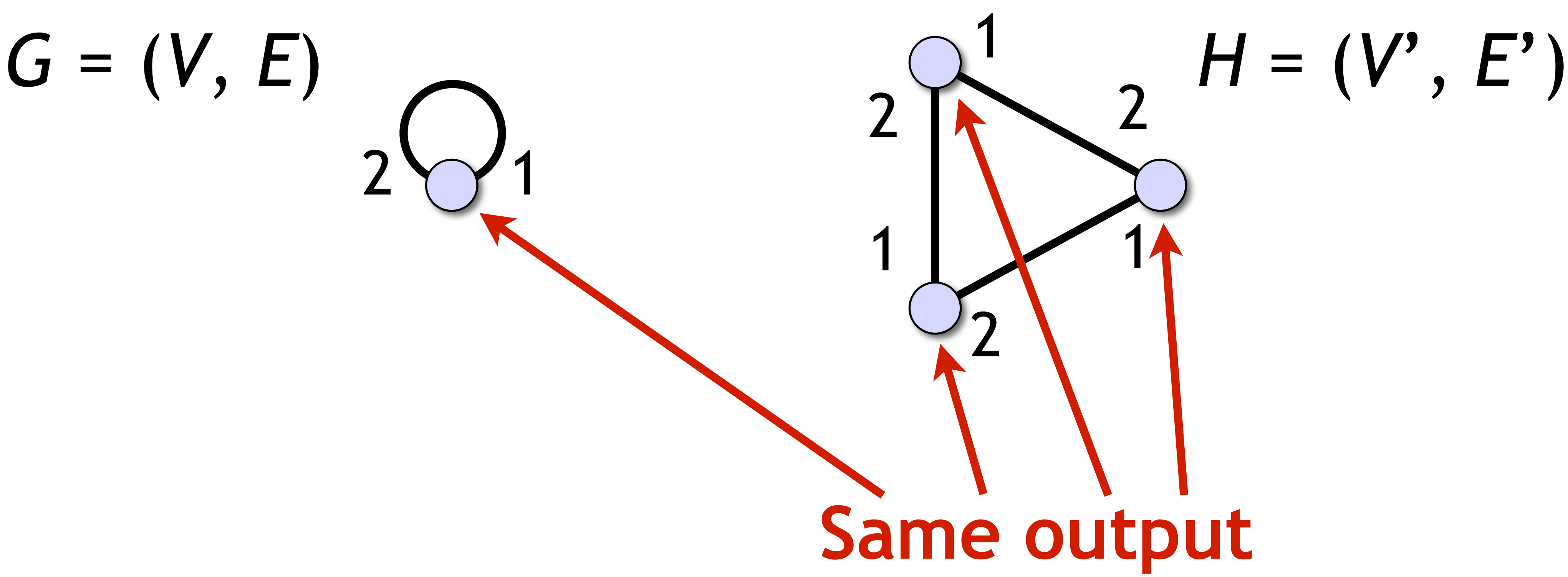


Covering maps and covering graphs

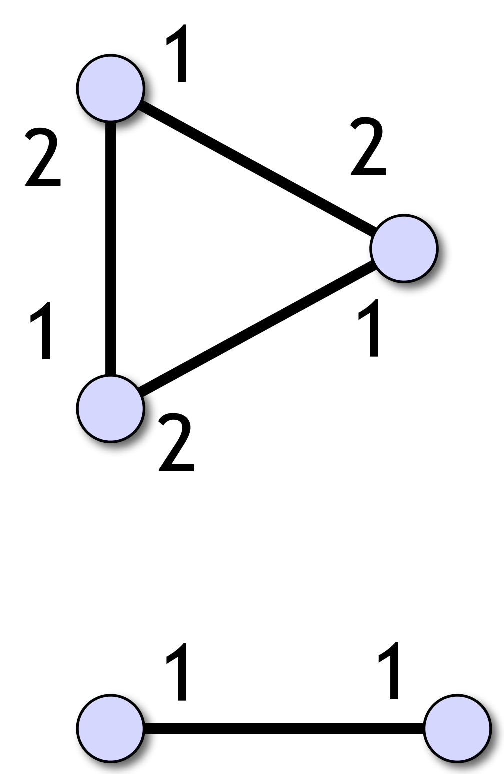


Covering maps and covering graphs

- Symmetric cycles are a simple special case of covering maps



Computability in the port-numbering model

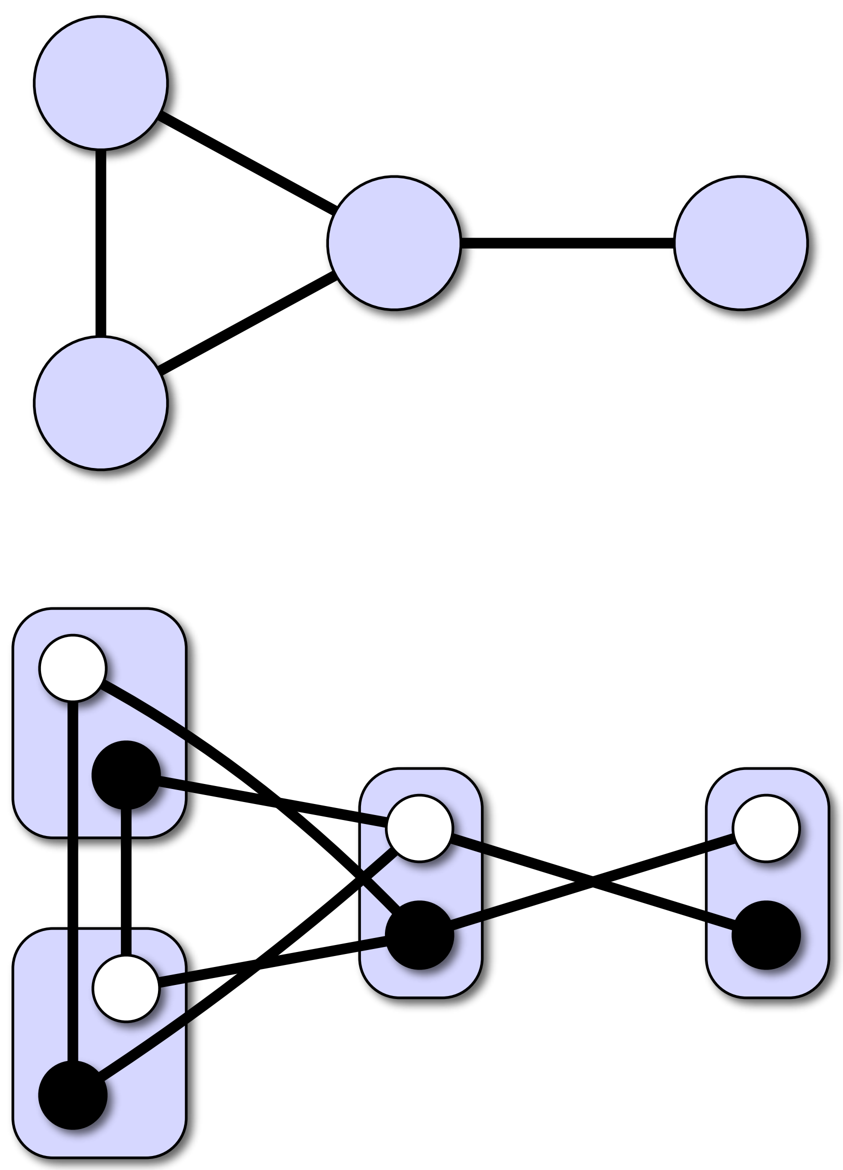


- Very limited model
  - in a cycle, we can only find a trivial solution: empty set, all nodes, ...
  - we can’t even break symmetry in a 2-node network!
- **What can be solved**?

Part III: Algorithms in port-numbering model

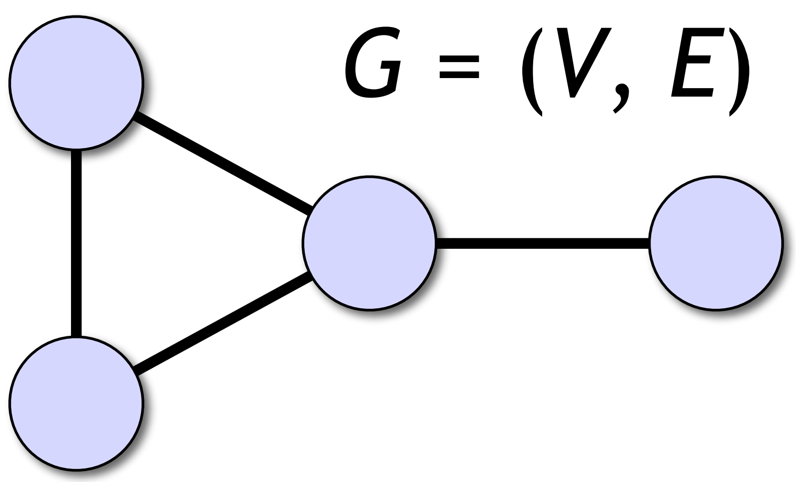
- Some problems *can* be solved in the port-numbering model...
  - and covering graphs can be used as an algorithm design technique, too!
- Example: vertex cover approximation

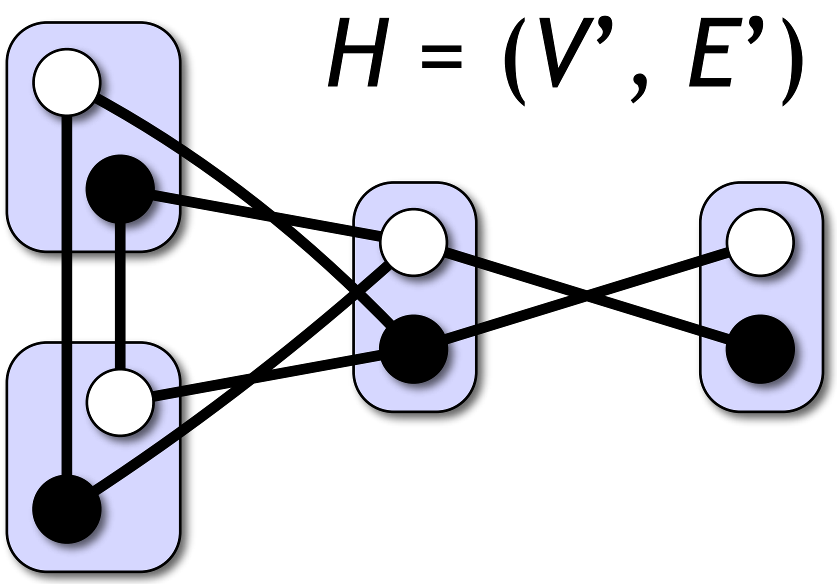
Symmetry breaking out of thin air: bipartite double covers



- Replace each node by **two virtual nodes**: black and white
  - original nodes **simulate** virtual nodes
  - each computers runs two programs in parallel: "black program" and "white program"
- Edges: **black-to-white**

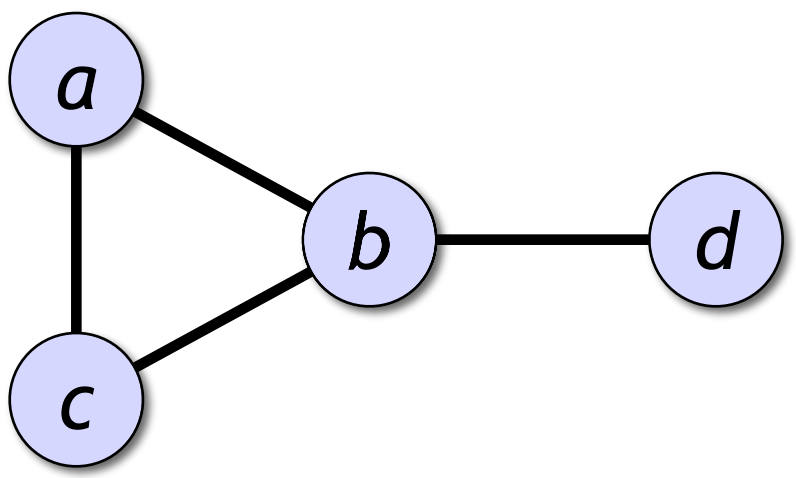
Symmetry breaking out of thin air: bipartite double covers

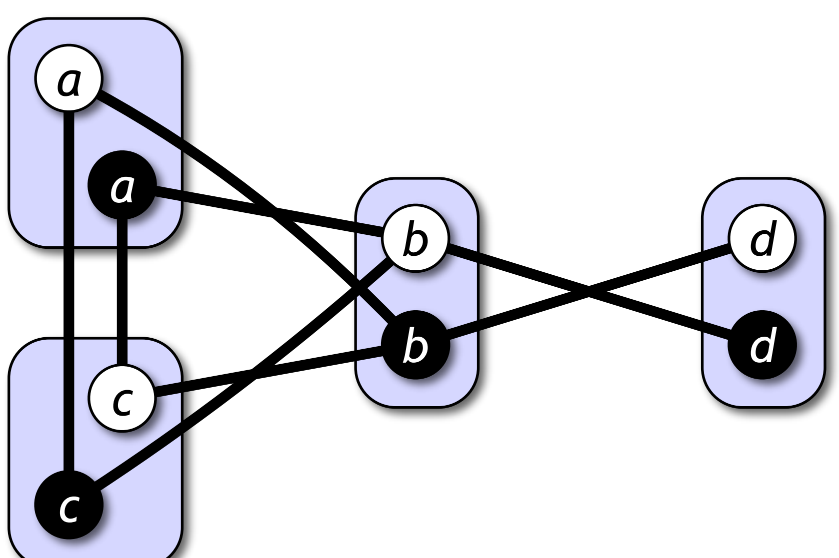




- Virtual graph *H* is a **covering graph** of *G*
- It is a **double** cover: 2 nodes of *H* map to each node of *G*
- It is **bipartite**
  - and we have already coloured its two parts: black and white!

Symmetry breaking out of thin air: bipartite double covers





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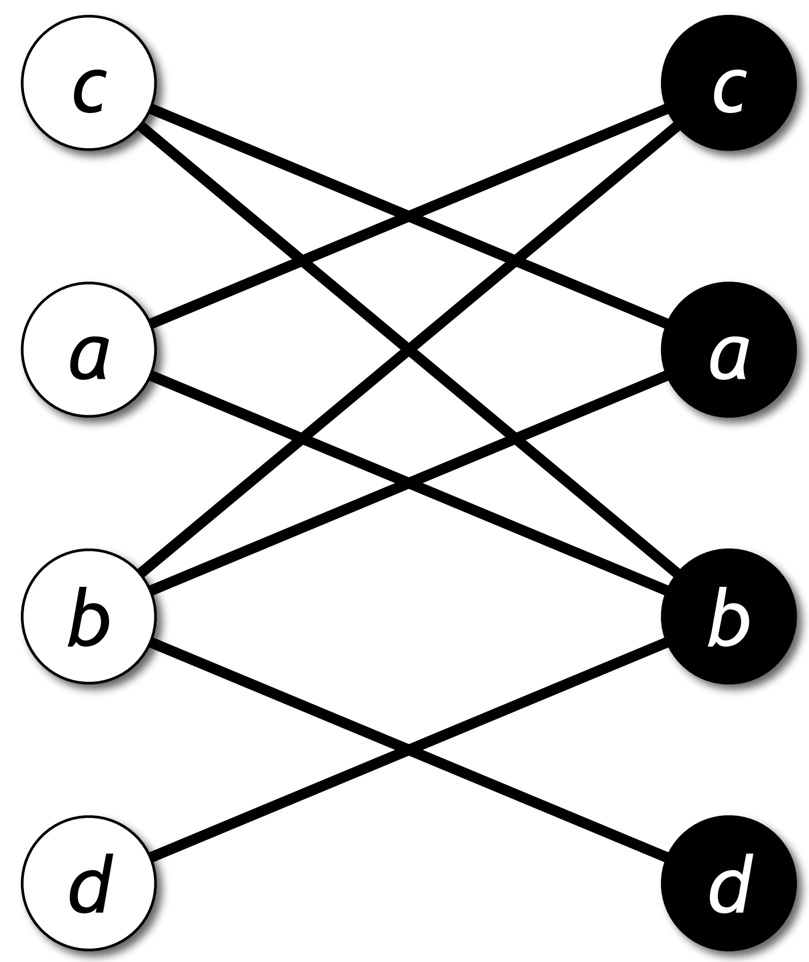
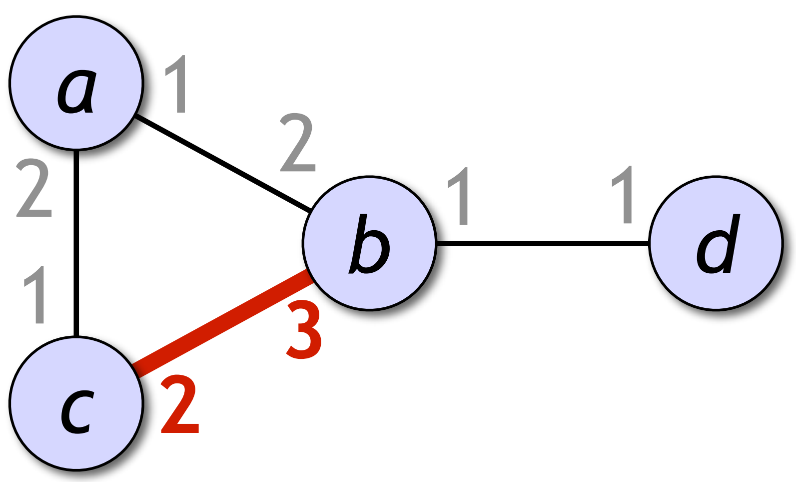
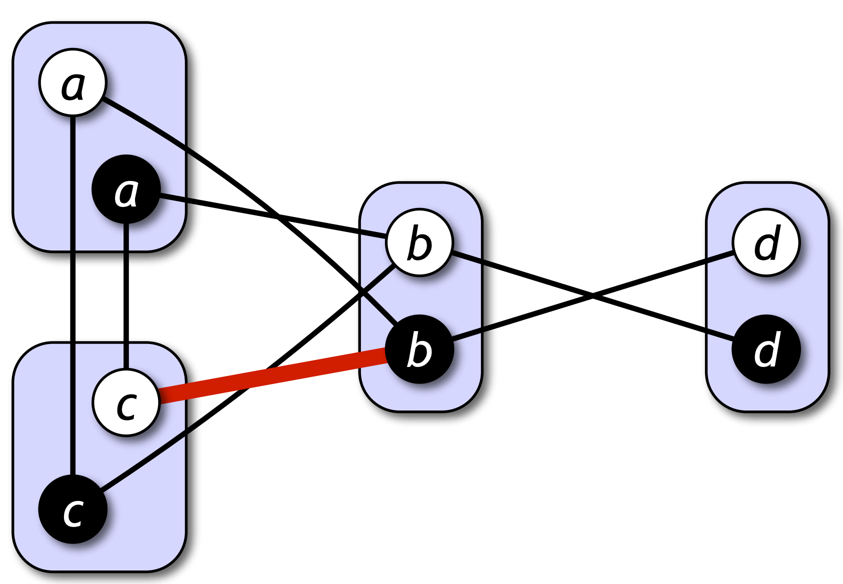
2-coloured graph



Symmetry breaking out of thin air: bipartite double covers





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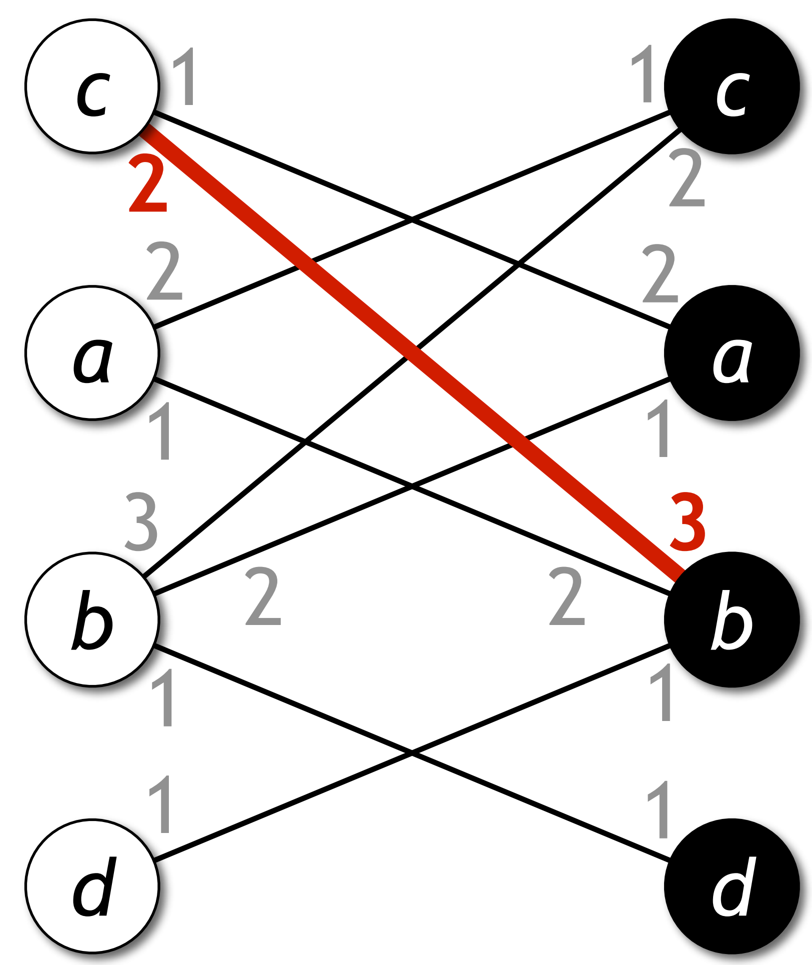
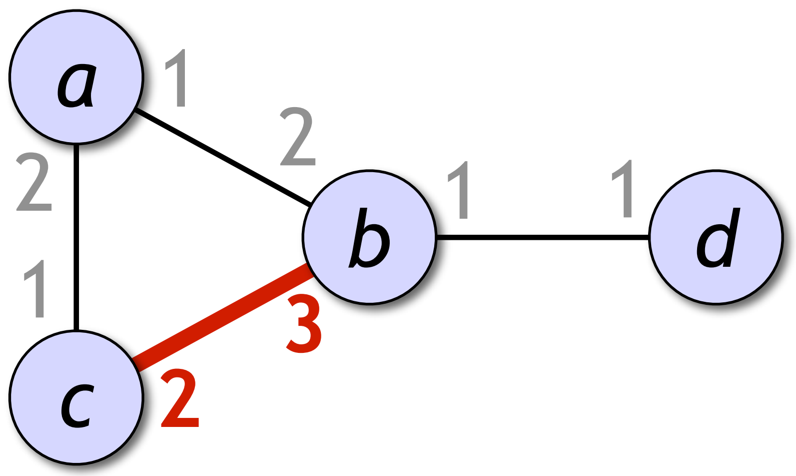
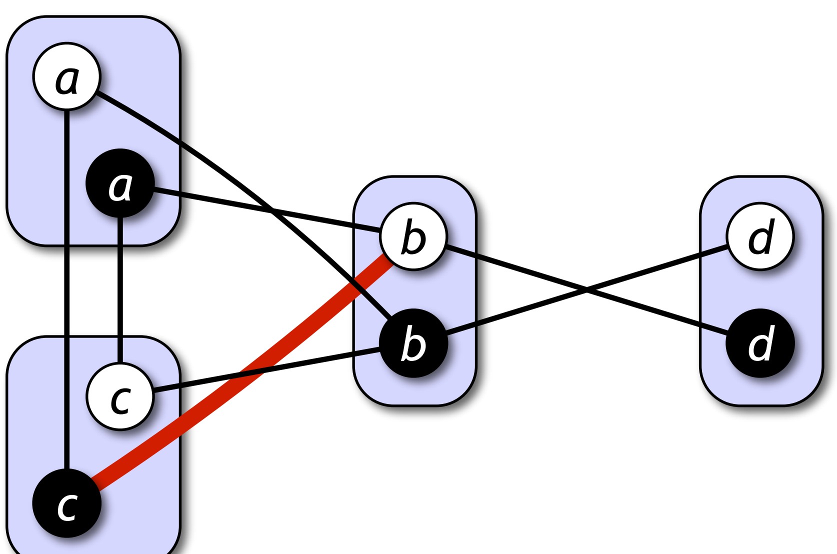
Port-numbering inherited



Symmetry breaking out of thin air: bipartite double covers





Port-numbering inherited

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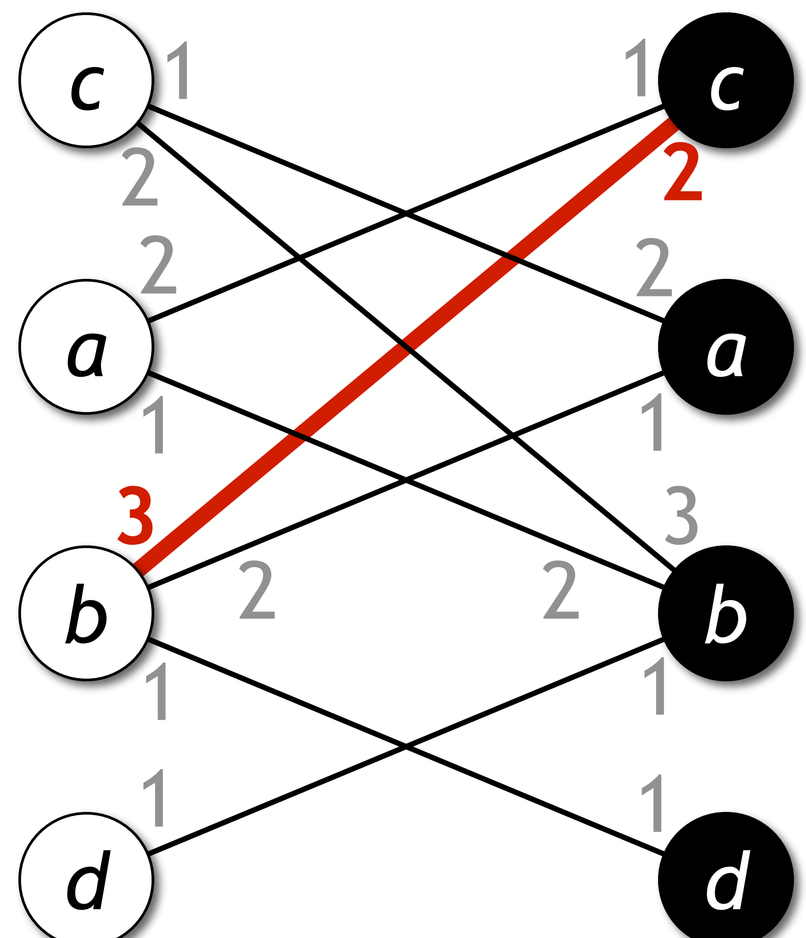
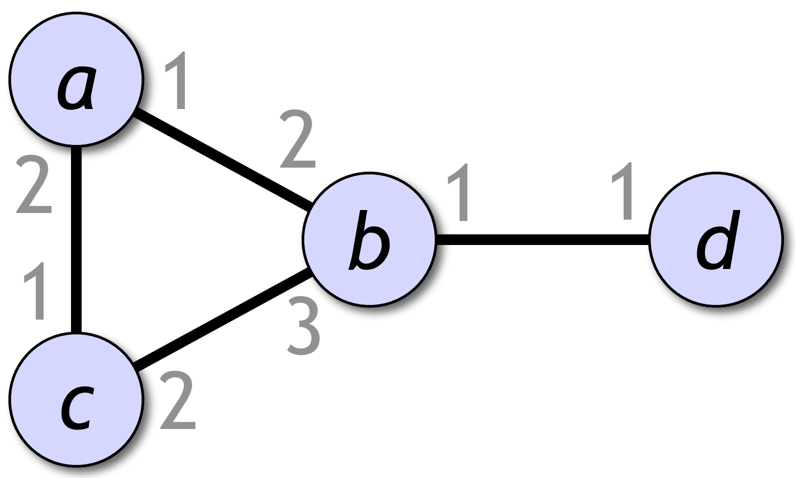


Symmetry breaking out of thin air: bipartite double covers



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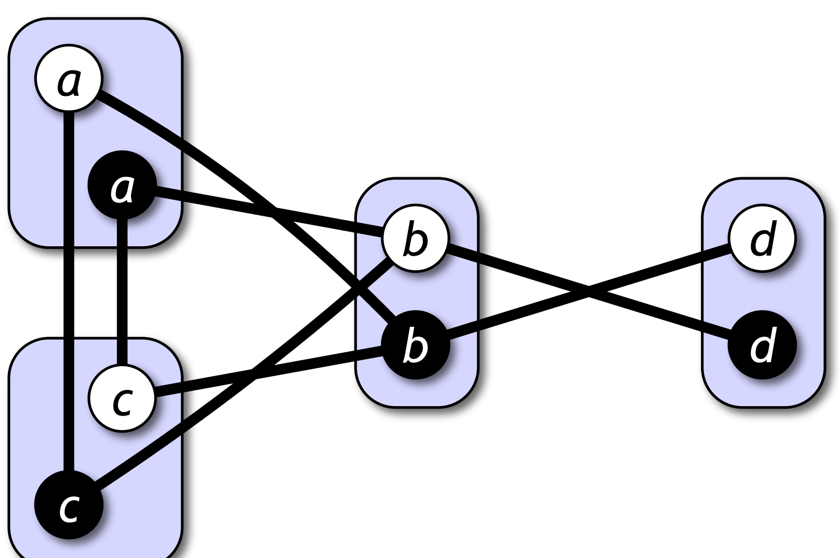

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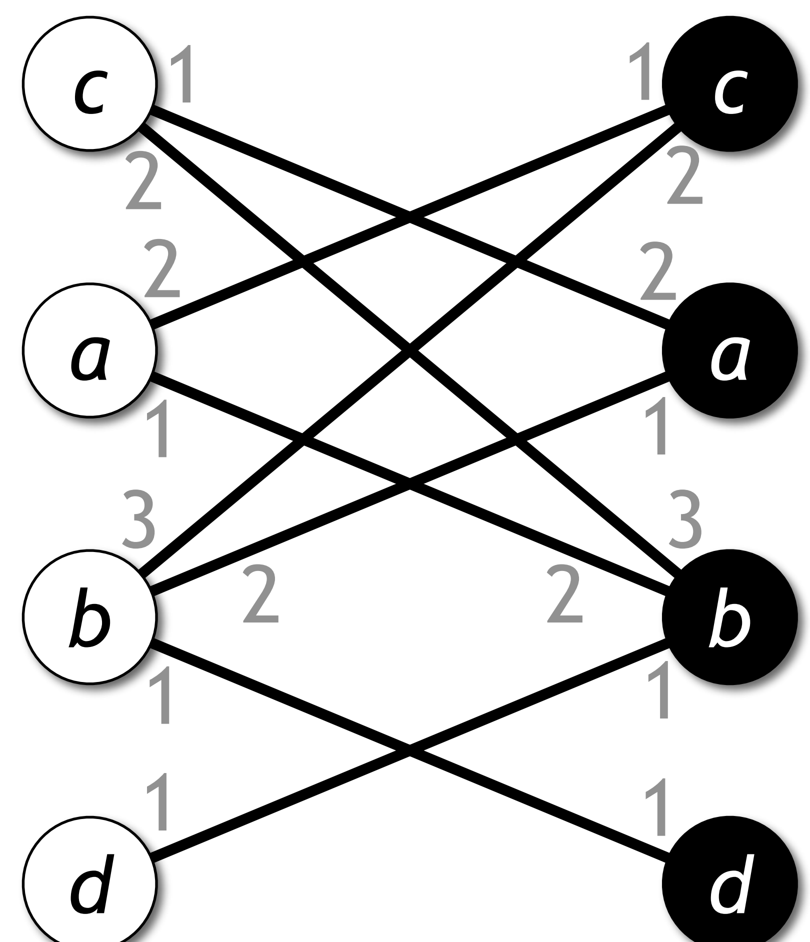
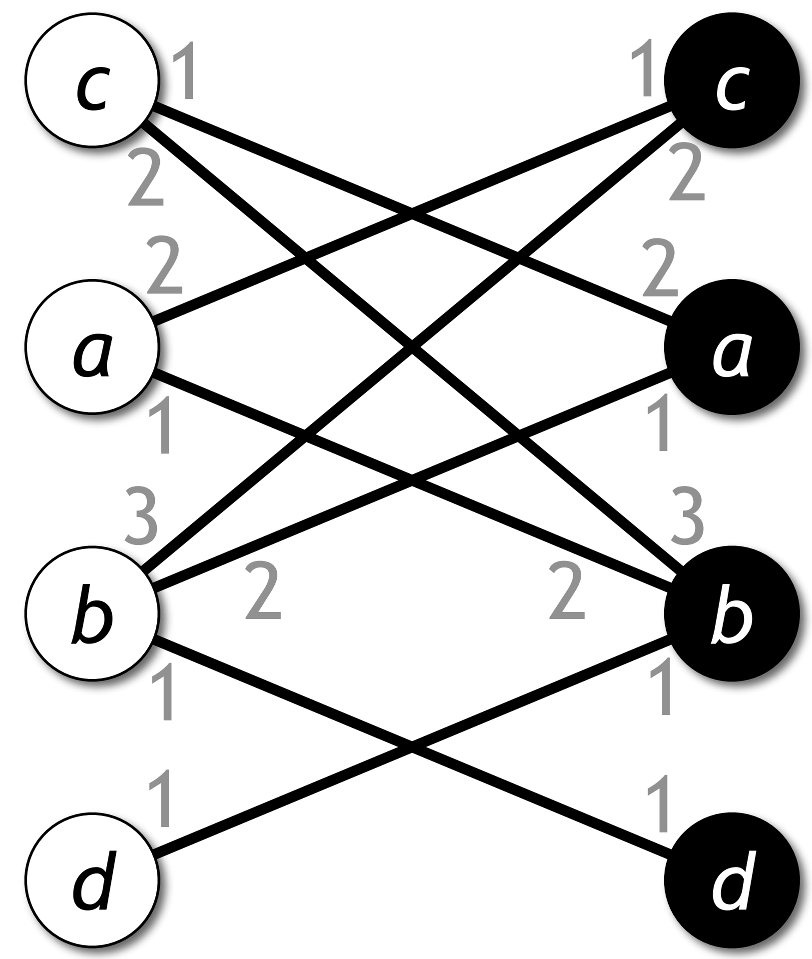


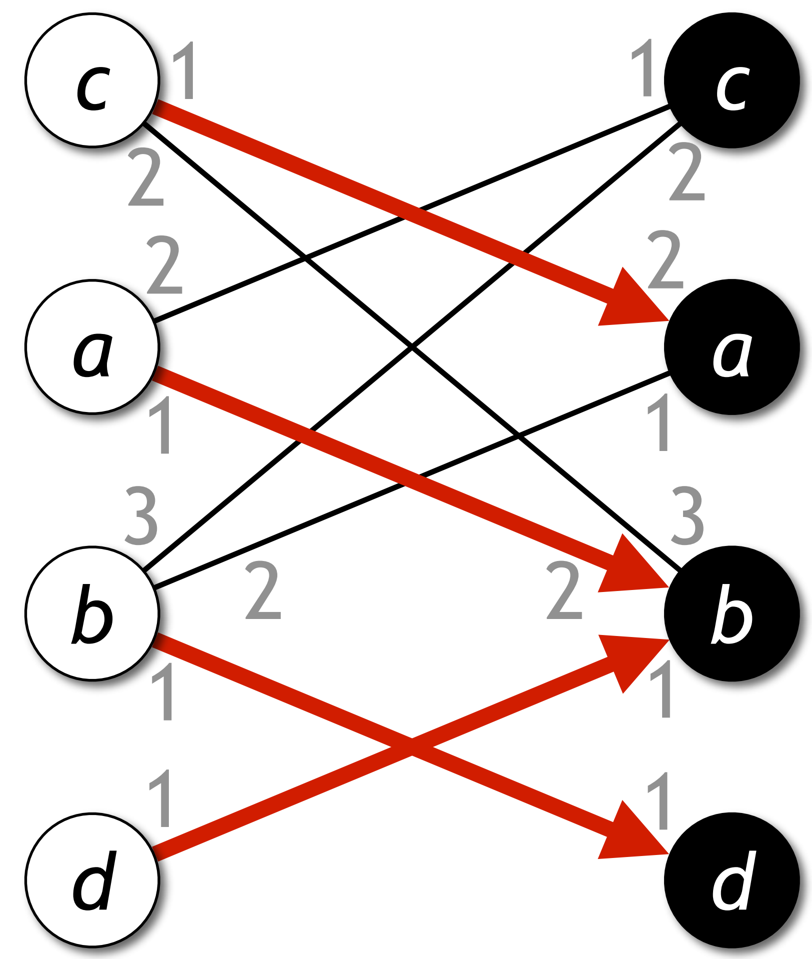
Symmetry breaking out of thin air: bipartite double covers

- Port-numbered graphs without colouring:
  - not possible to find a maximal matching (consider an even cycle)
- Port-numbered graphs with 2-colouring:
  - very easy to find a maximal matching!



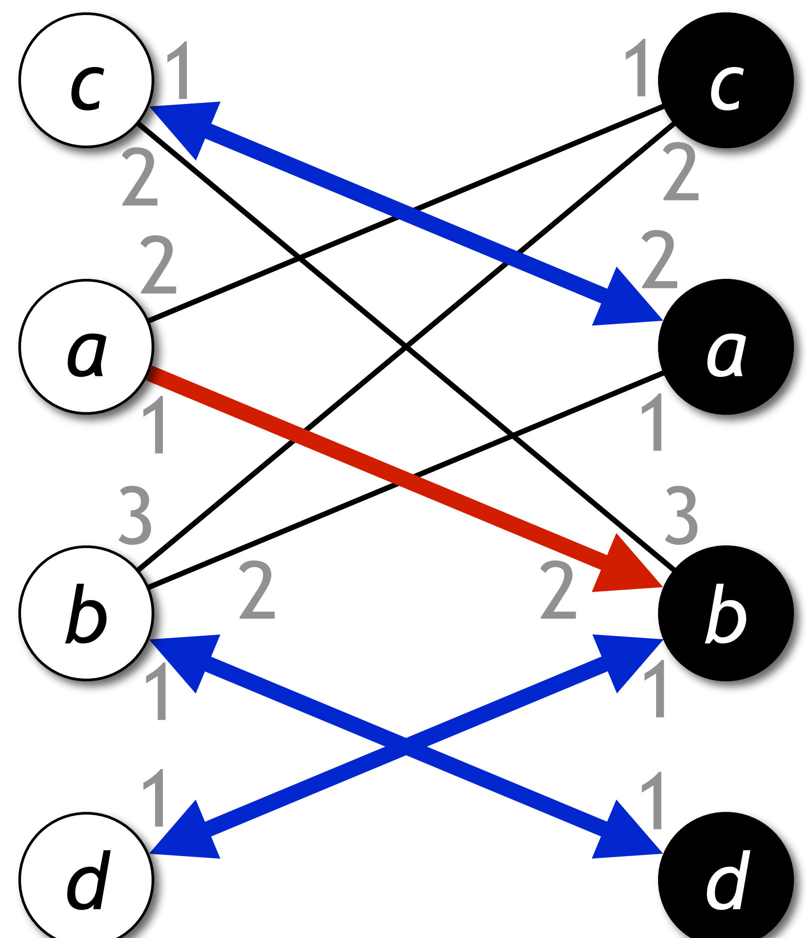
Maximal matching in 2-coloured graphs

- Each white node sends **proposals** to its black neighbours
  - one by one,
    order by port numbers



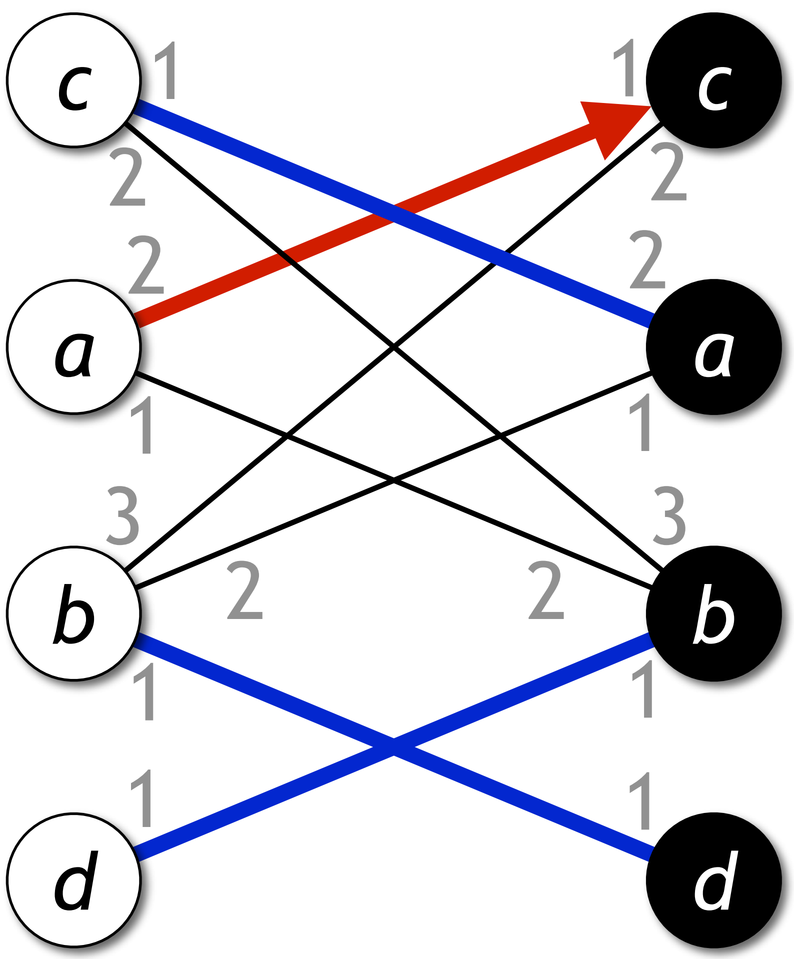
Maximal matching in 2-coloured graphs

- Each white node sends **proposals** to its black neighbours
  - one by one, order by port numbers
- Each black node **accepts** the first proposal it gets
  - break ties using port numbers



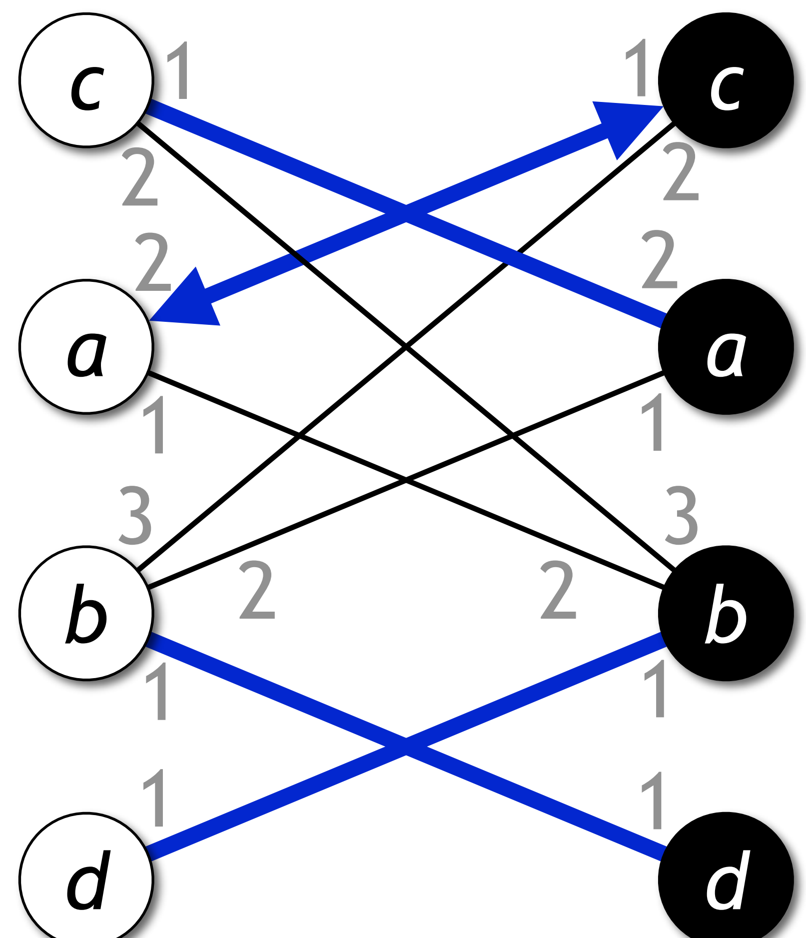
Maximal matching in 2-coloured graphs

- Each white node sends **proposals** to its black neighbours
  - one by one,
    order by port numbers
  - until its proposal
    is accepted, or
    all neighbours
    have rejected



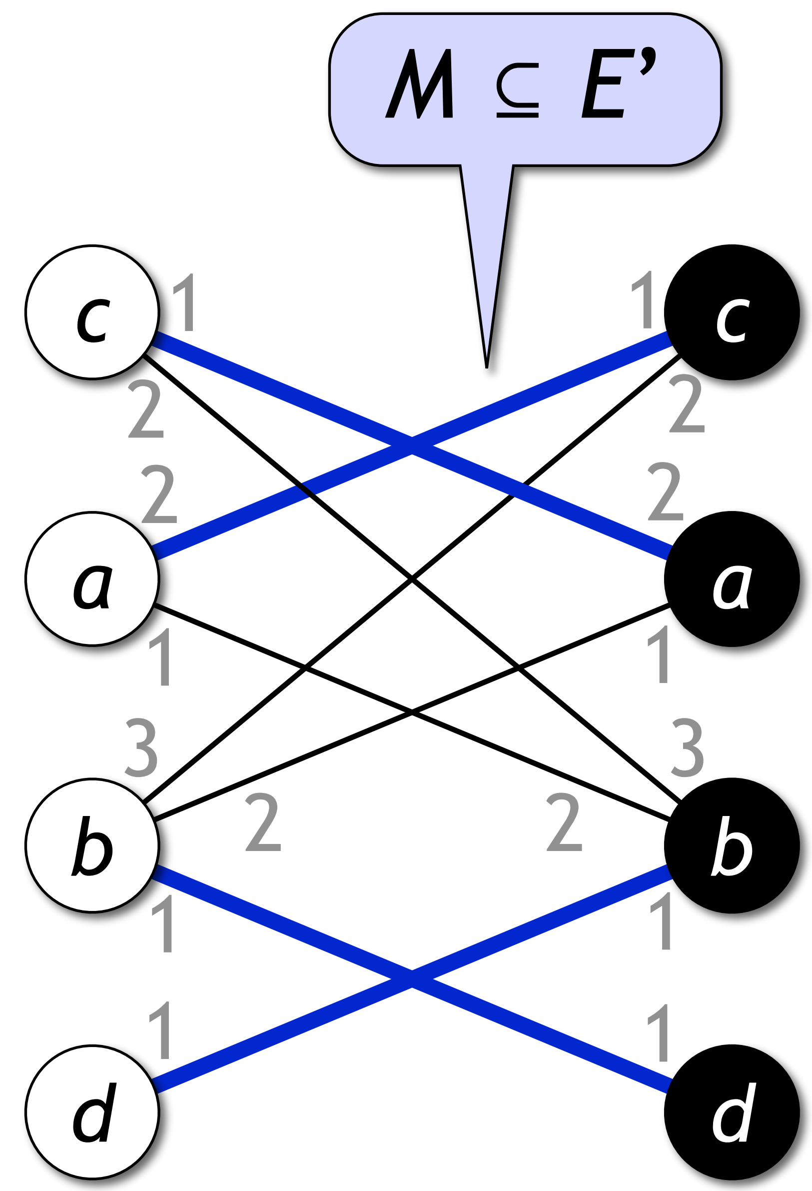
Maximal matching in 2-coloured graphs

- Each white node sends **proposals** to its black neighbours
  - one by one,
    order by port numbers
- Each black node **accepts** the first proposal it gets
  - break ties using
    port numbers



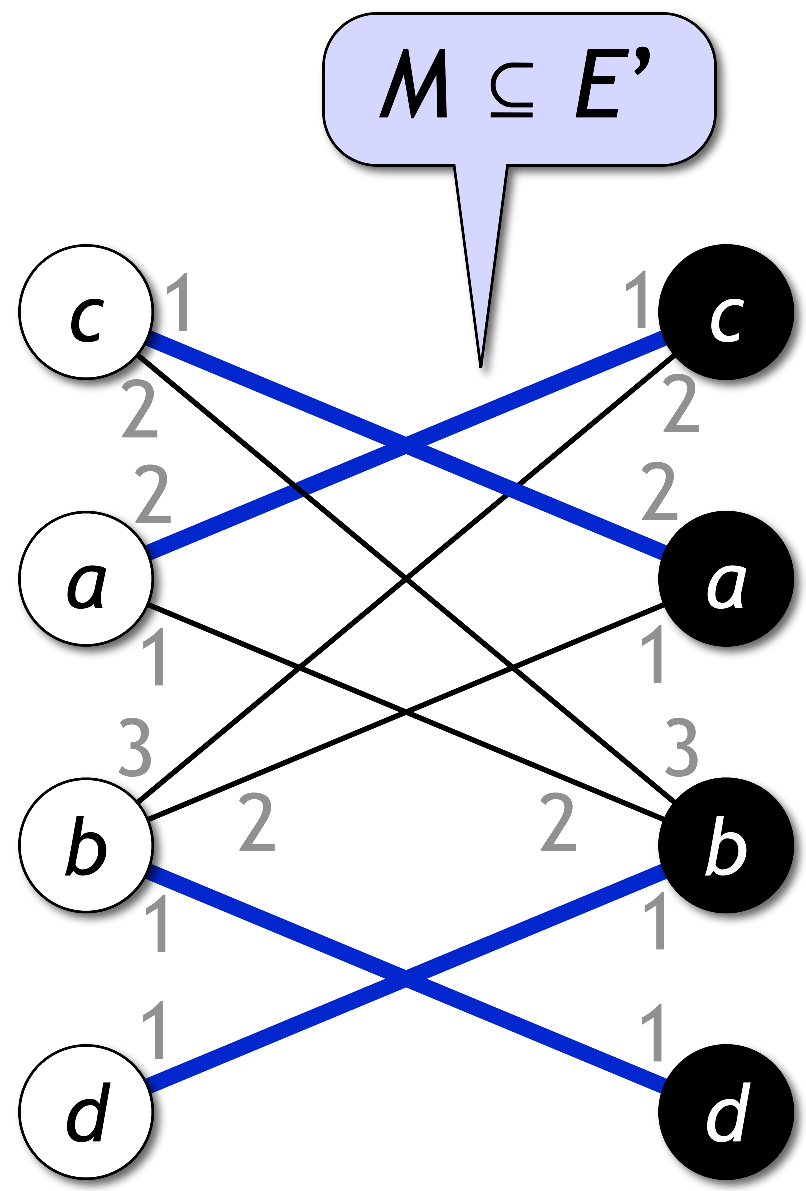
Maximal matching in 2-coloured graphs

- Accepted proposals *M*: **matching**
  - white nodes don't propose after acceptance
  - black nodes don't accept more than once
  - all nodes incident to at most one edge

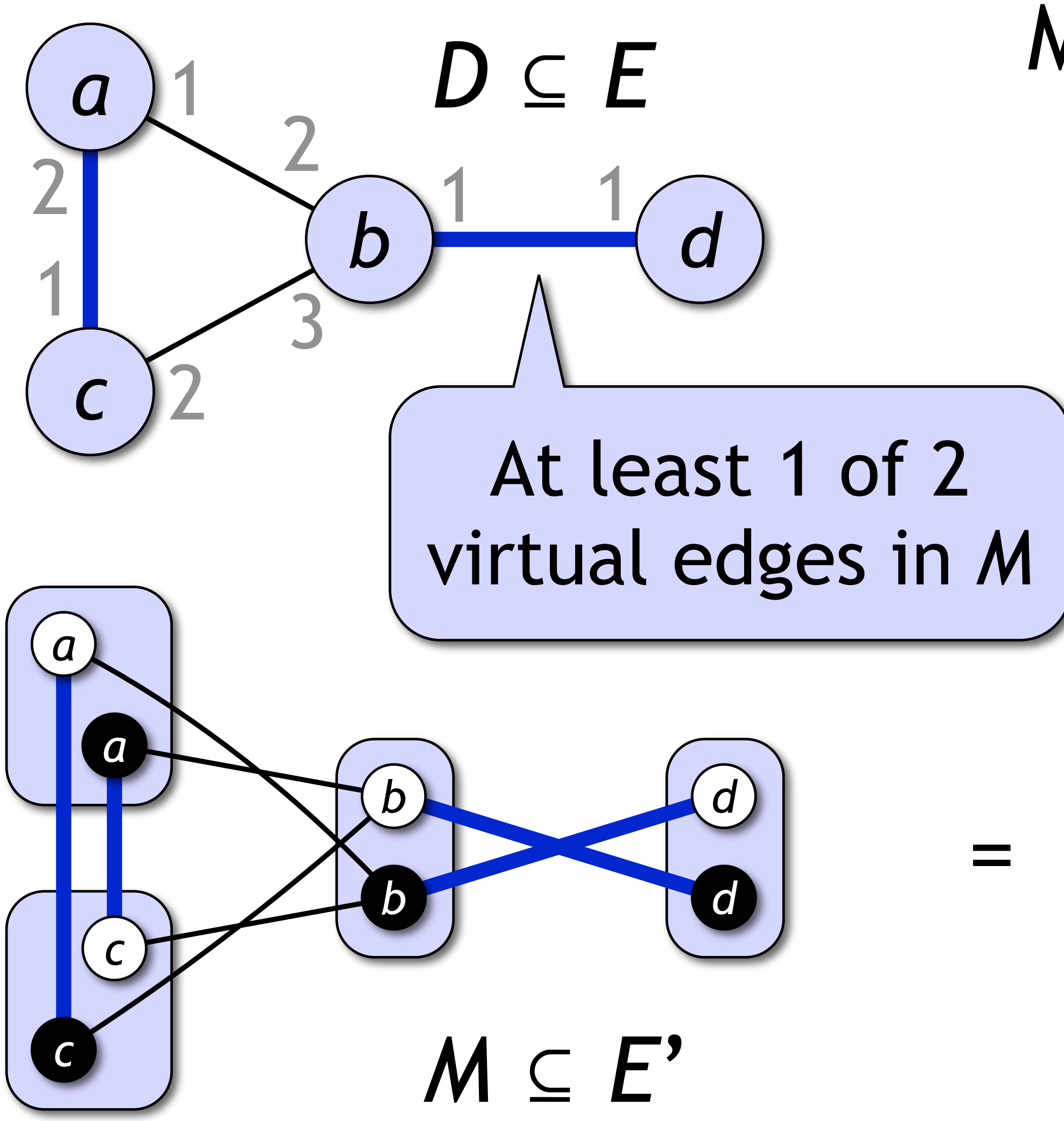


Maximal matching in 2-coloured graphs

- Accepted proposals *M*: **maximal matching**!
  - assume $\{u, v\} \in E \setminus M$
 u unmatched
  - then *u* has sent a proposal to *v* and *v* has rejected it
  - therefore *v* had already received another proposal, v is matched
  - can't add $\{u, v\}$ to *M*



Maximal matching in bipartite double cover



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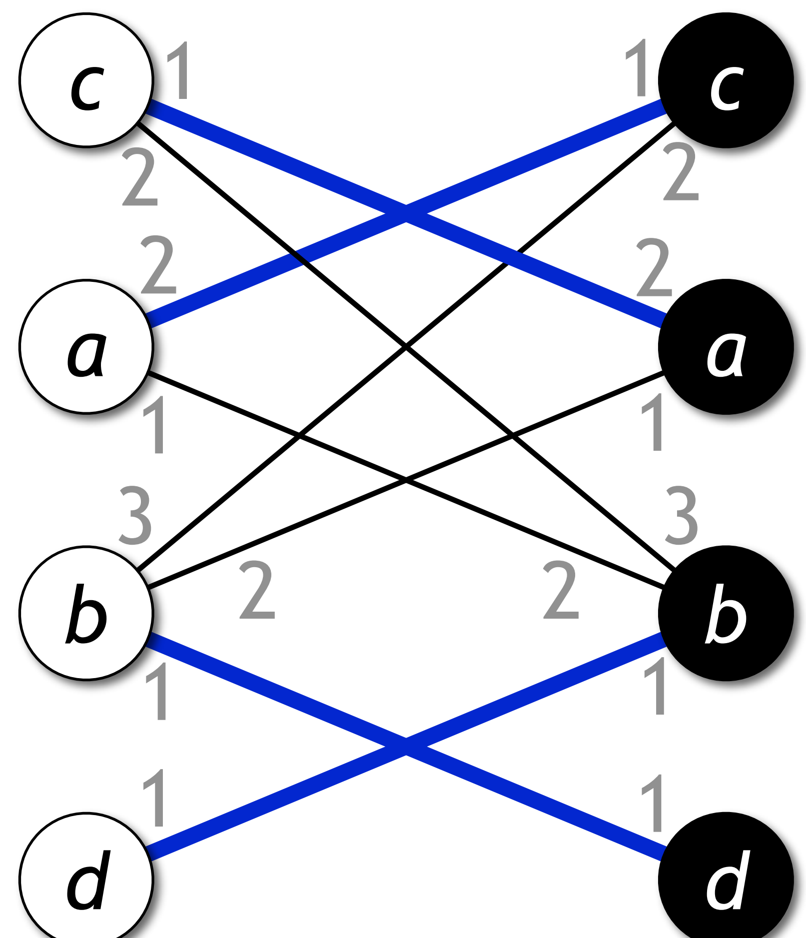
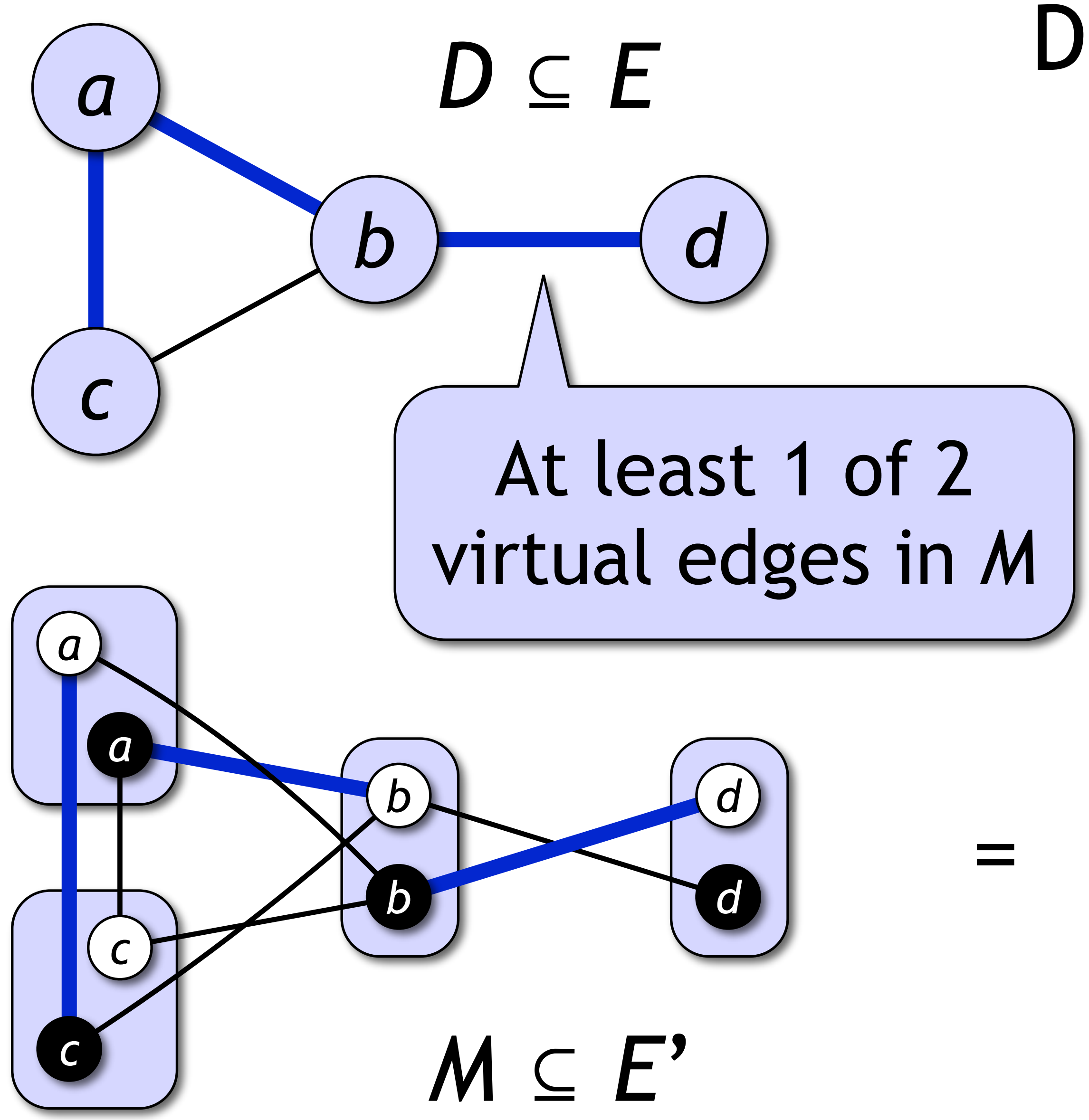
Map back to original graph



Maximal matching in bipartite double cover

Different possibilities...



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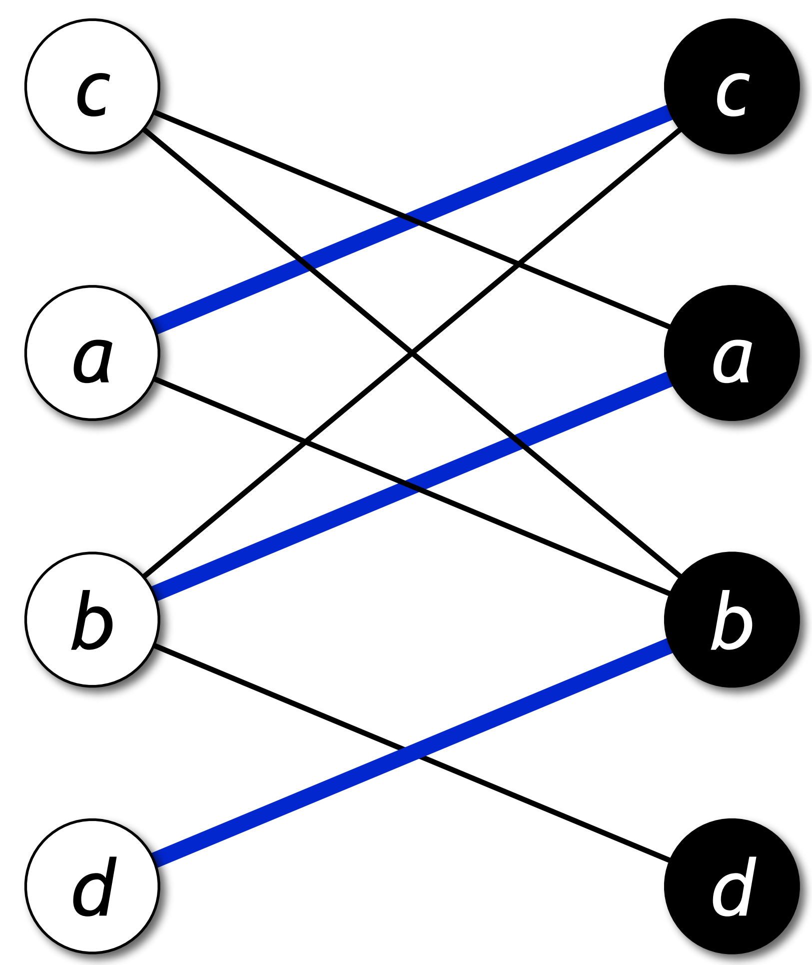
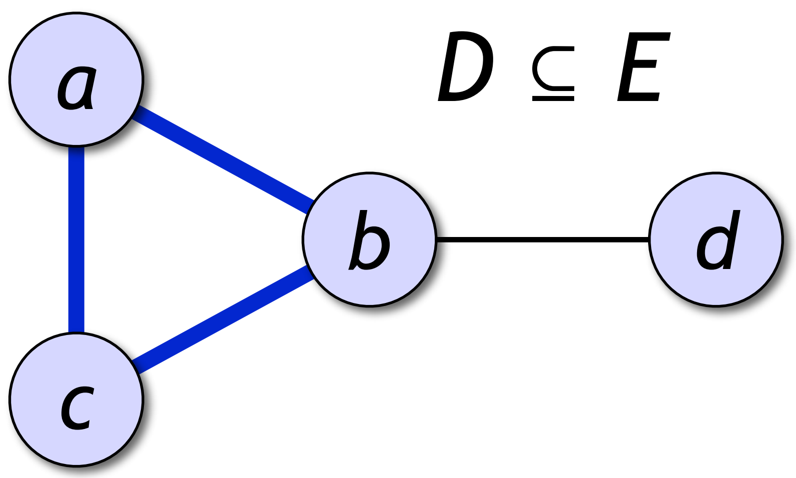
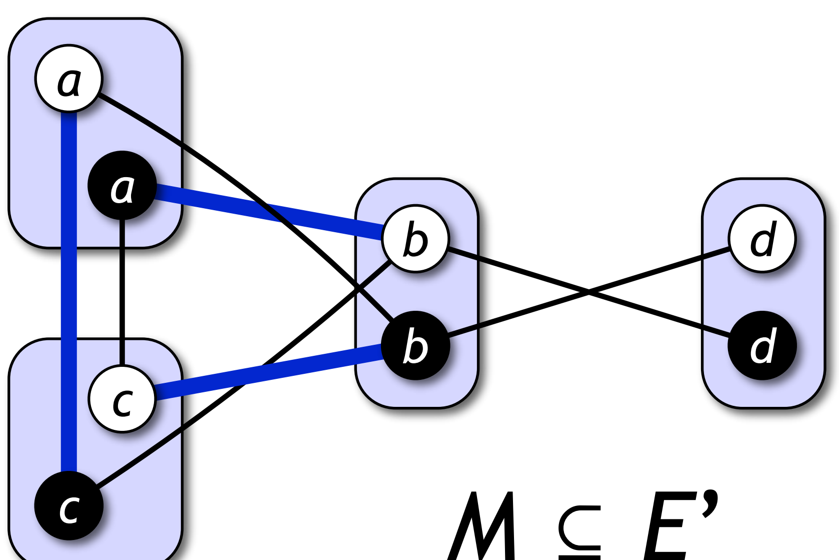


Maximal matching in bipartite double cover



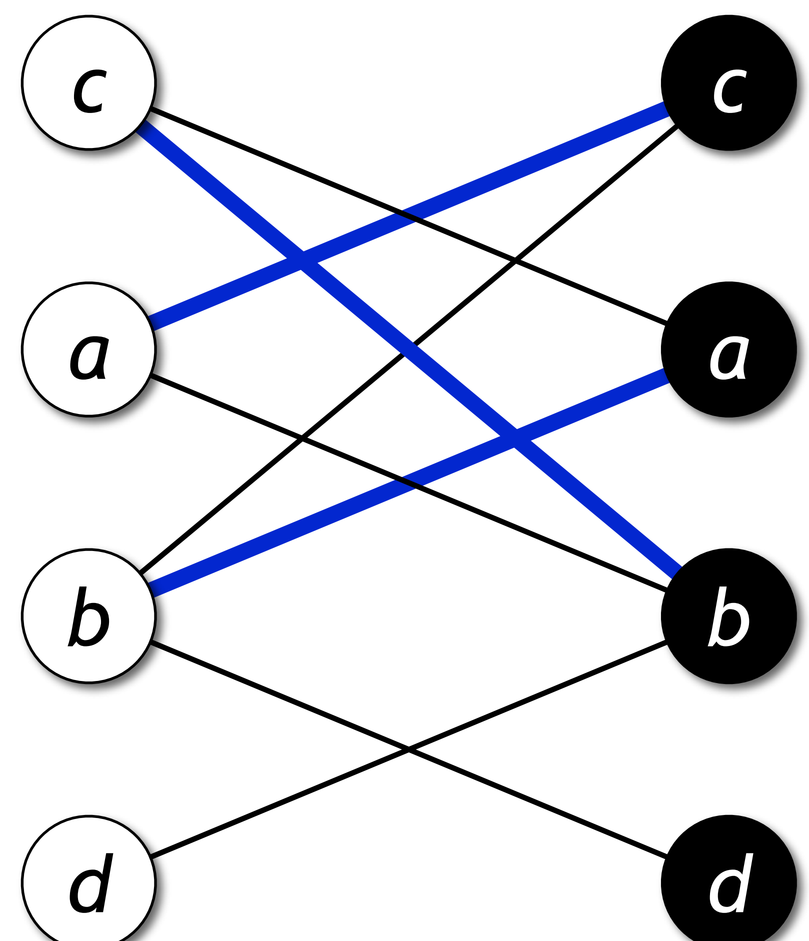
$D \subseteq E$



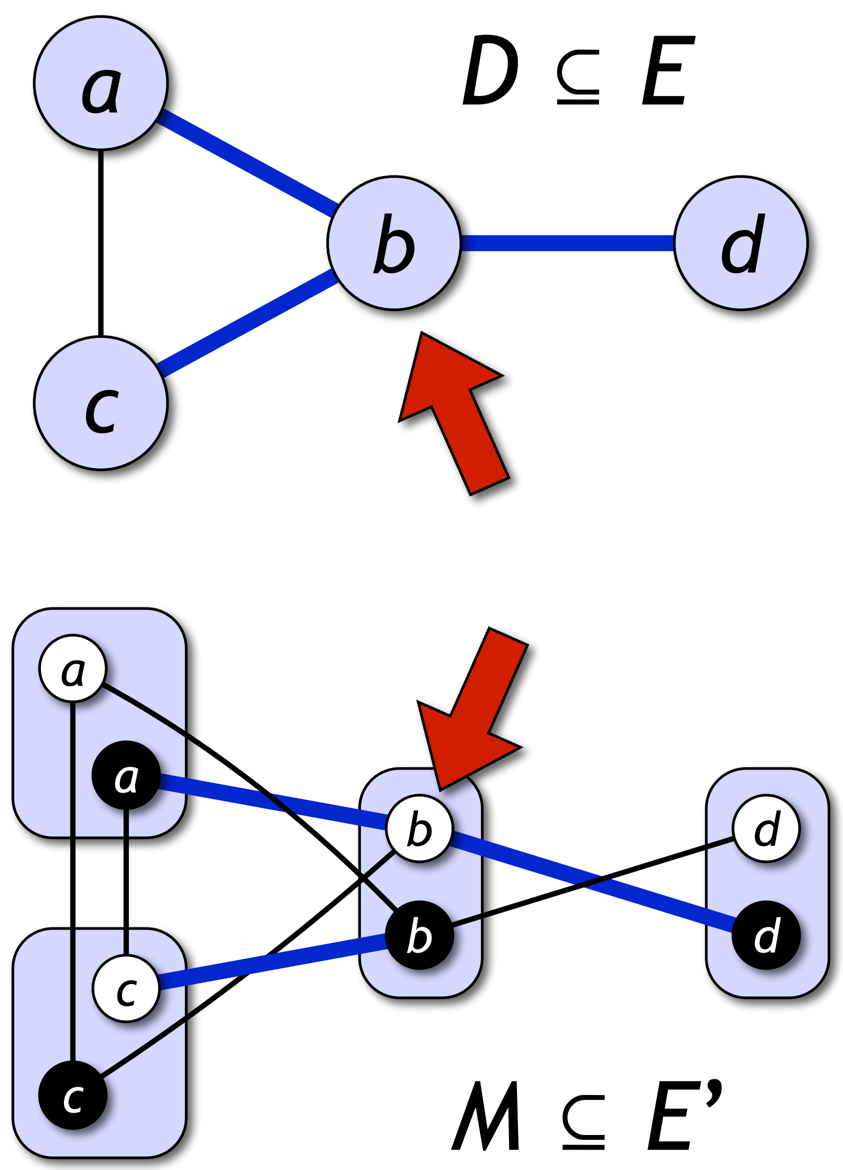
$M \subseteq E'$

=

Different possibilities...

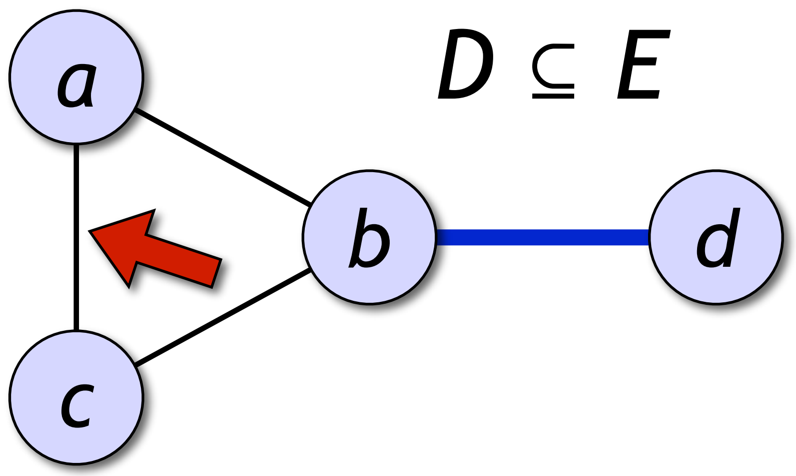


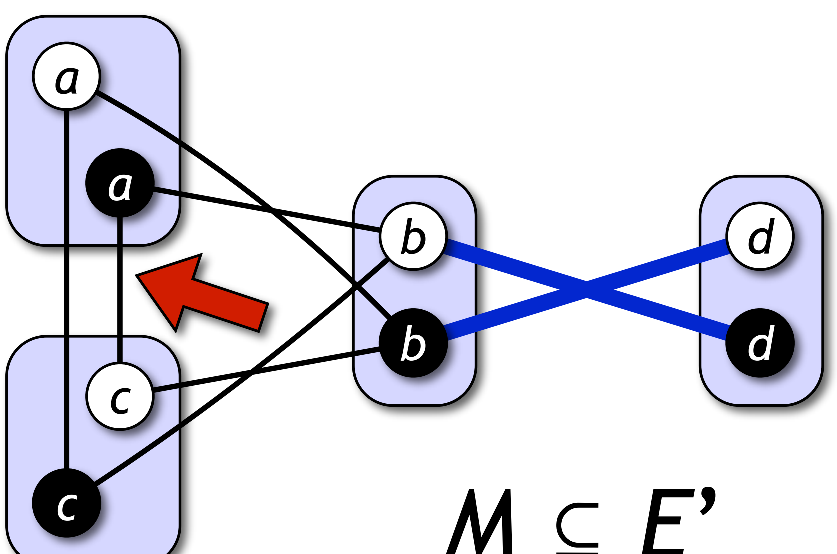
Maximal matching in bipartite double cover



- However, this is not possible, because M is a matching
  - M induces a subgraph of H with max. degree **1**
  - therefore: D induces a subgraph of G with max. degree **2**

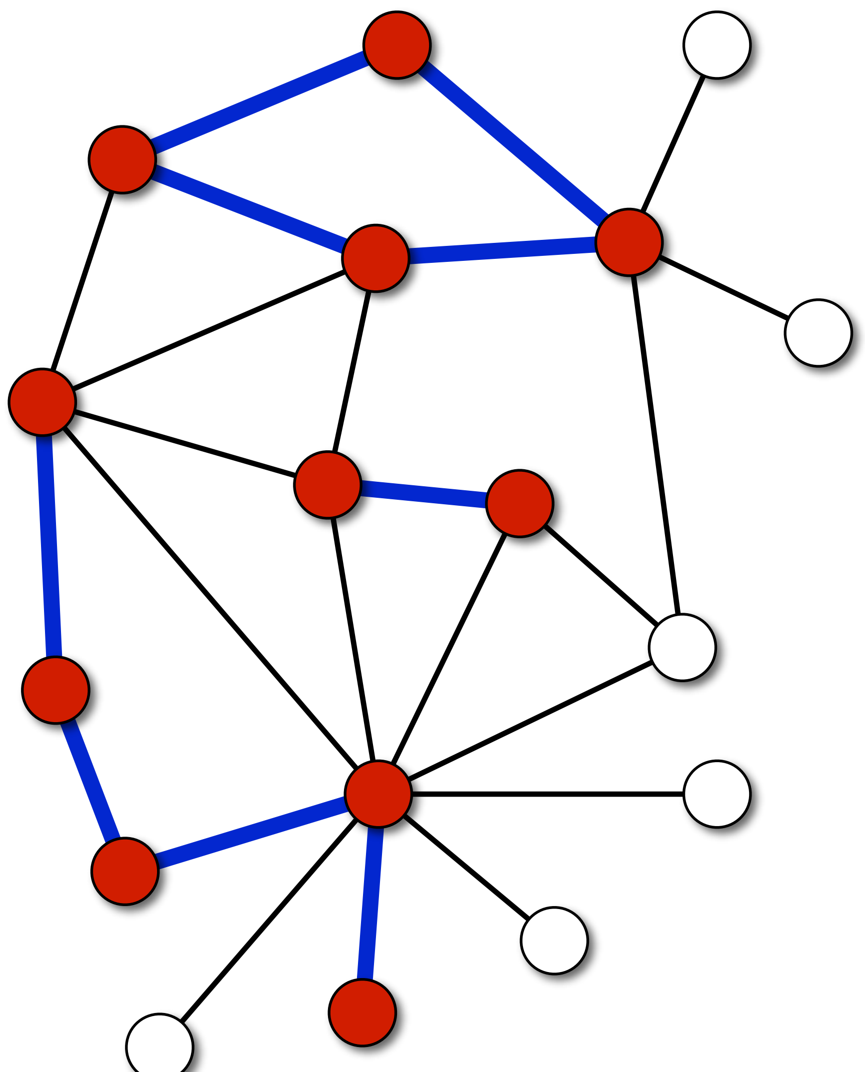
Maximal matching in bipartite double cover





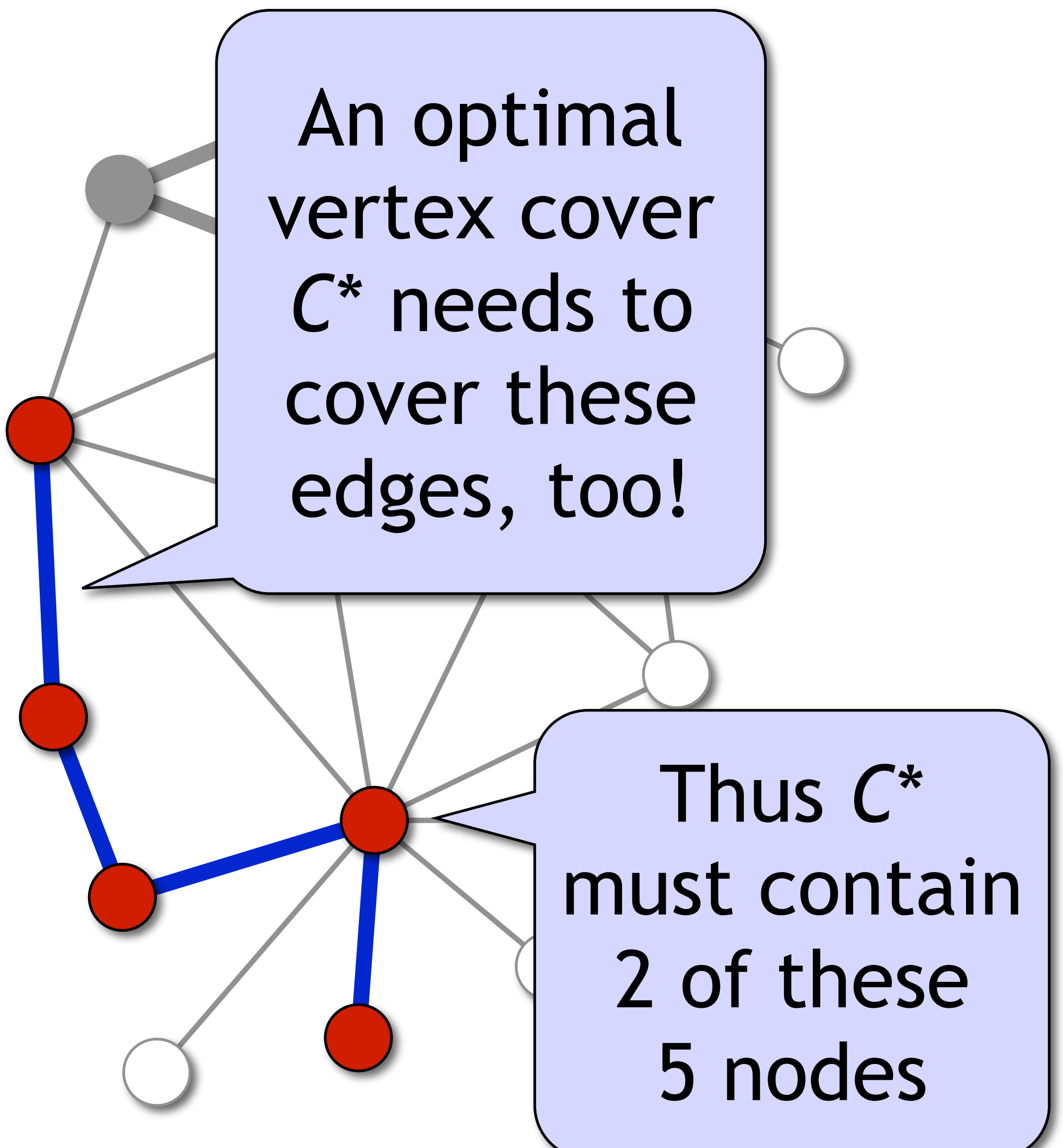
- And this is not possible, because M is maximal
  - each edge of H is in M or shares at least one endpoint with M
  - endpoints of M form a **vertex cover** in H
  - endpoints of D form a **vertex cover** in G!

Finding a vertex cover



- So we will find a set *D* of edges such that:
  - *D* induces a subgraph of maximum degree 2
  - *D* must consist of **paths** and **cycles**
  - endpoints of *D* form a **vertex cover** *C*
  - is it a small vertex cover?

Finding a vertex cover



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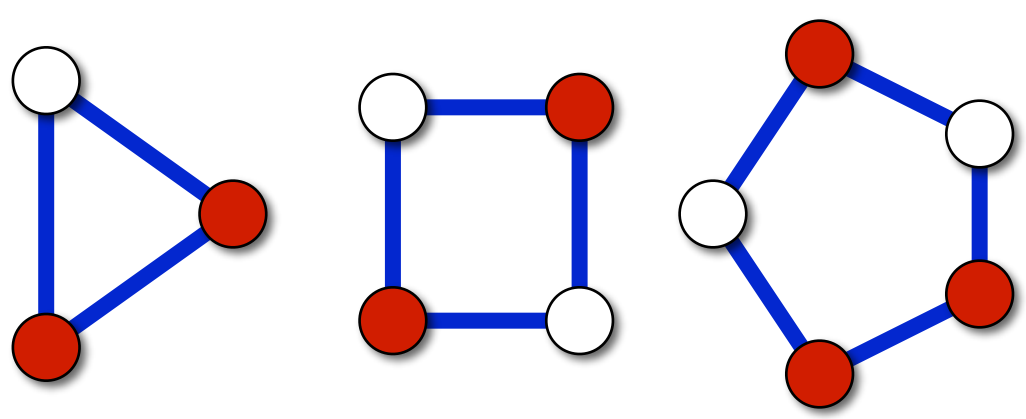
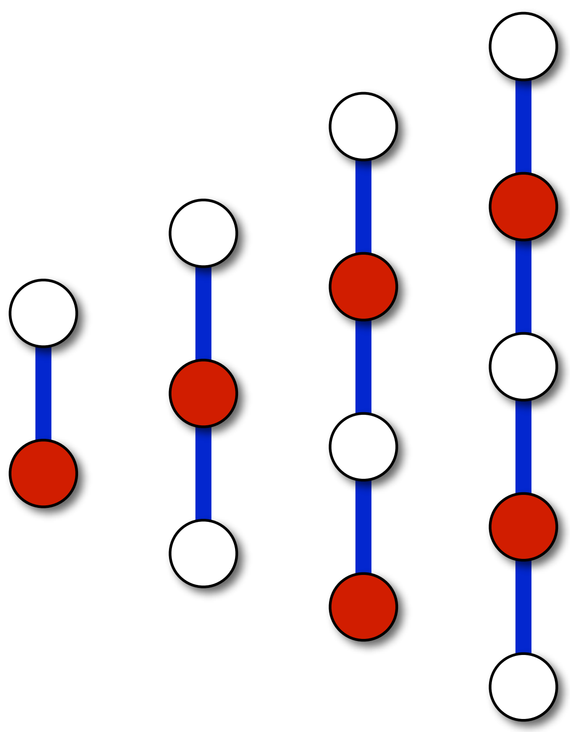
Finding a vertex cover



- Different cases:
  - Cycle with 3 edges:
    3 nodes in C, ≥ 2 in C^*
  - Cycle with 4 edges:
    4 nodes in C, ≥ 2 in C^*
  - Cycle with 5 edges:
    5 nodes in C, ≥ 3 in C^*
    ...

$|C| \leq 2|C^*|$

Finding a vertex cover

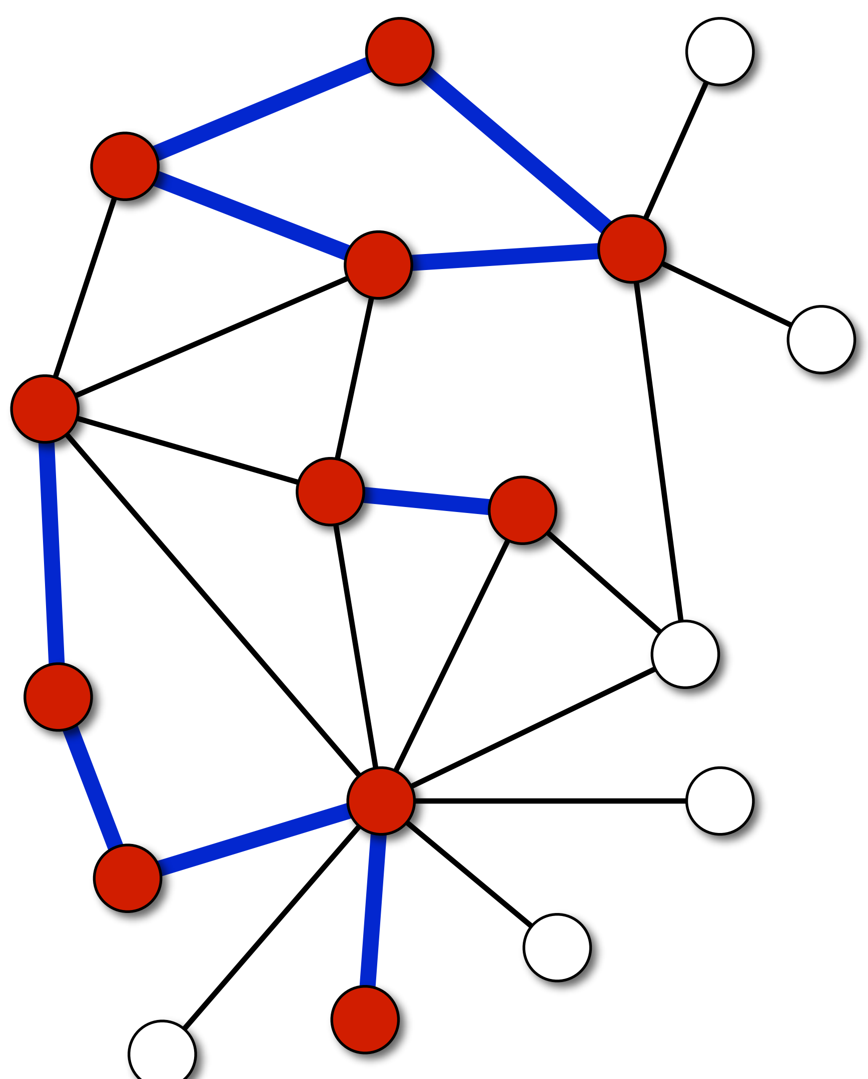


$|C| \leq 3|C^*|$

- Different cases:
  - Path with 1 edge:
    2 nodes in C, $\geq$ 1 in C^*
  - Path with 2 edges:
    3 nodes in C, $\geq$ **1** in C^*
  - Path with 3 edges:
    4 nodes in C, $\geq$ 2 in C^*
  - Path with 4 edges:
    5 nodes in C, $\geq$ 2 in C^*

    …

Finding a vertex cover



- In each path or cycle:
    - C has at most **3** times as many nodes as C^*
- Summing over all paths and cycles:
    - $|C| \leq \mathbf{3}|C^*|$
- The algorithm finds a **3-approximation** of minimum vertex cover!

Finding a vertex cover: summary

- Vertex cover is a graph problem that *can* be solved reasonably well in the **port-numbering model** with a deterministic distributed algorithm
  - And the algorithm was simple and fast: $O(\Delta)$ rounds! (here Δ = maximum degree)
- Coming next month: how to find a **2-approximation** of vertex cover in $O(\Delta)$ rounds

Finding a vertex cover: two very different worlds

- Centralised setting, polynomial-time algorithms:
  - **trivial** to find a *minimal vertex cover*: greedy algorithm
  - it requires more thought to find a good *approximation of minimum vertex cover*
- Distributed setting, port-numbering model:
  - **impossible** to find a *minimal vertex cover*: symmetry breaking issues
  - but we have seen that it is possible to find a good *approximation of minimum vertex cover*

Summary

- Deterministic distributed algorithms
  - Synchronous communication rounds
  - Port-numbering model
- Covering maps and covering graphs
  - Technique for proving negative results:
    these nodes will always produce the same output
  - Algorithm design technique:
    bipartite double covers, 2-colouring