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Locality

in online, dynamic, sequential, and distributed graph algorithms

Joint work with:

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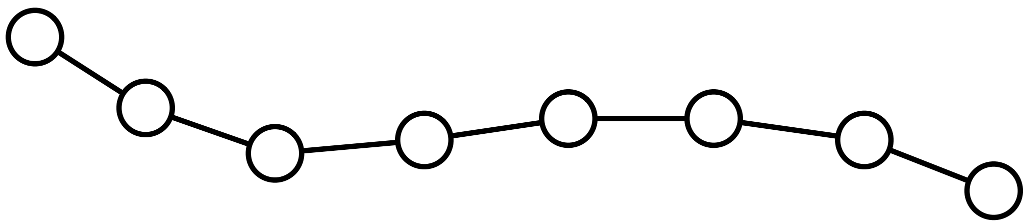
- Navid Eslami
- Henrik Lievonen
- Darya Melnyk
- Joona Särkijärvi

arxiv.org/abs/2109.06593

Informal introduction to locality

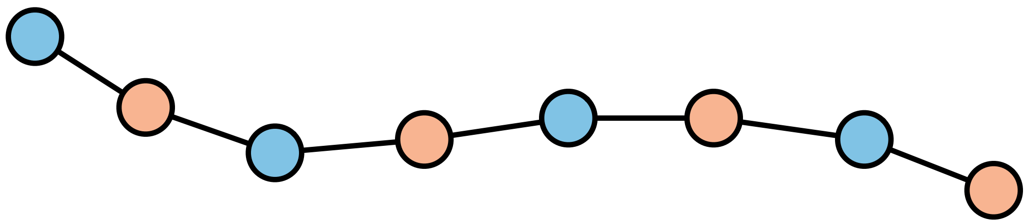
Coloring paths locally?

- **Given:** a path graph



Coloring paths locally?

- **Given:** a path graph
- **Find:** a proper 2-coloring



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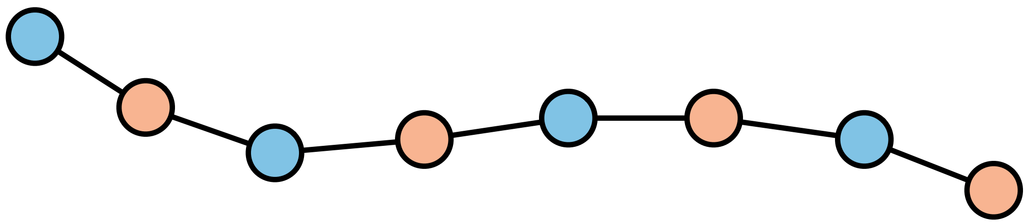

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- **Restriction:** the color of each node ***only depends on its local neighborhood***
  - say, radius-$o(n)$ neighborhood

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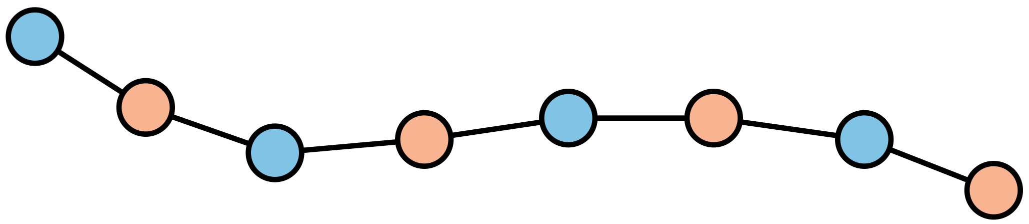

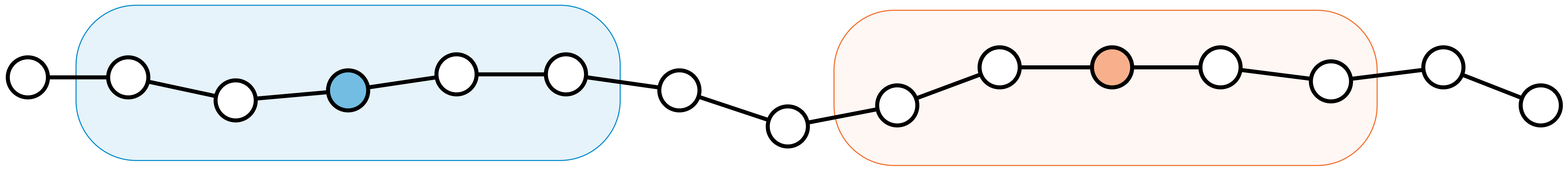

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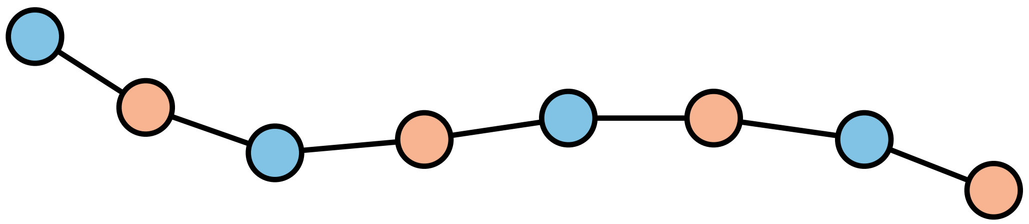

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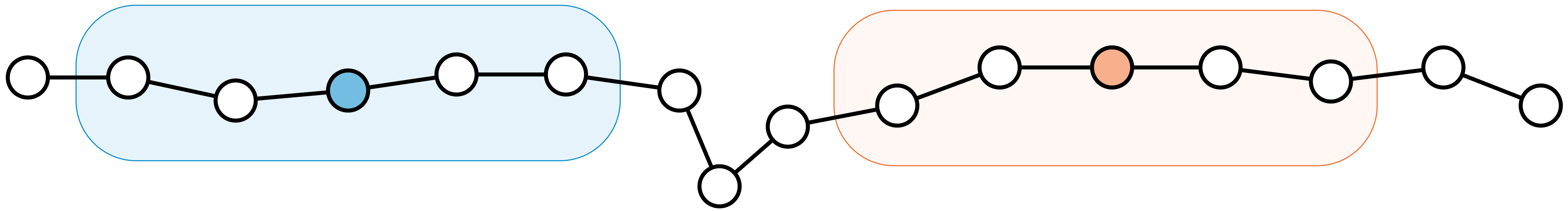
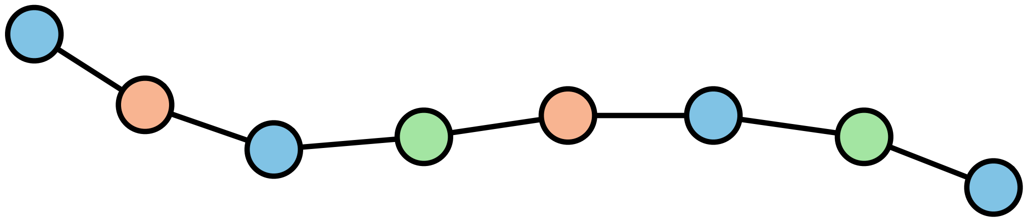


Conclusion:

Paths can't be 2-colored with any local strategy

… and it doesn't really depend on exactly how we define "local"

3-coloring paths locally?



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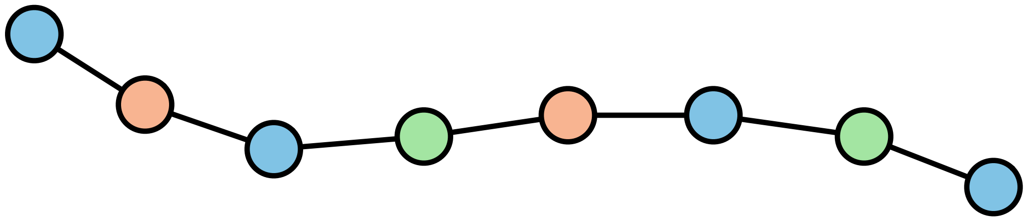
3-coloring paths locally?



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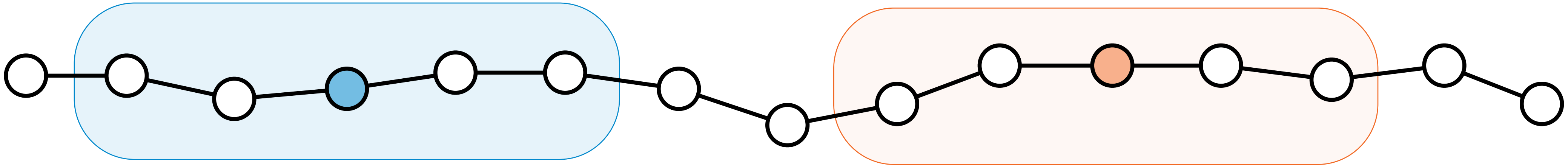

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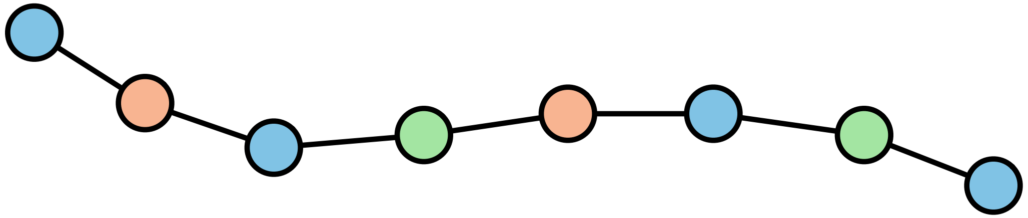

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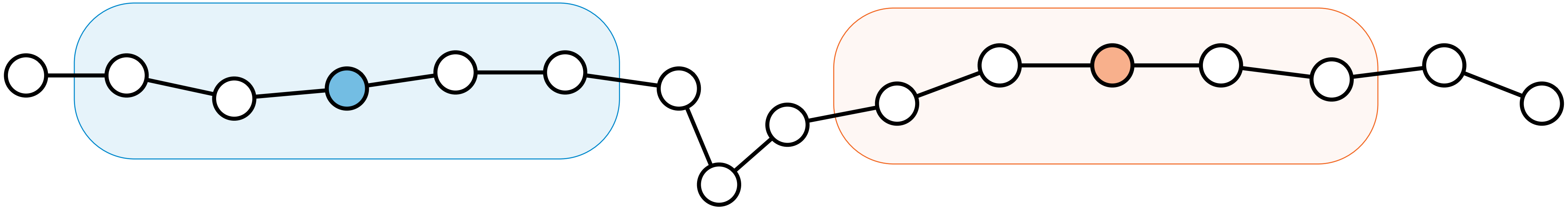

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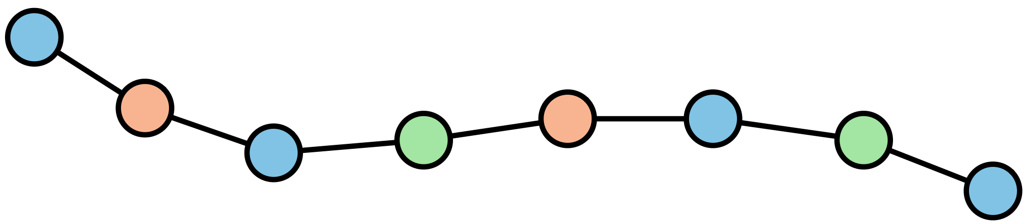

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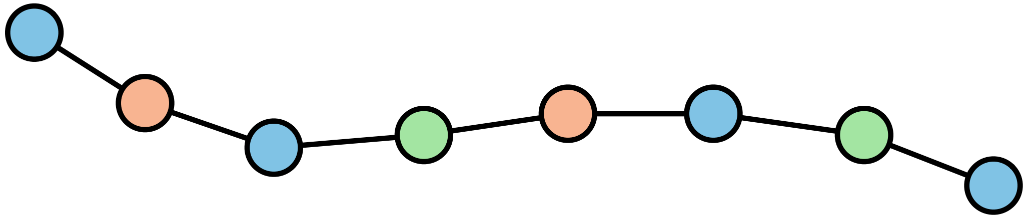

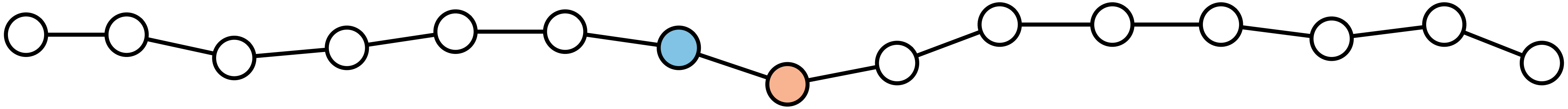

Conclusion:

We need some way to break local symmetry

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We need some way to break local symmetry

- randomness
- unique node identifiers
- sequential ordering …

3-coloring paths locally

Nodes labeled with (small) unique identifiers:

locality ≈ ½ log* *n*

3-coloring paths locally

Nodes labeled with (small) unique identifiers:

locality ≈ ½ log* n

Nodes labeled with random bit strings:

locality ≈ ½ log* n

3-coloring paths locally

Nodes labeled with (small) unique identifiers:

$$\textbf{locality} \approx \tfrac{1}{2} \log^* n$$

Nodes labeled with random bit strings:

$$\textbf{locality} \approx \tfrac{1}{2} \log^* n$$

[Cole & Vishkin 1986, Linial 1992, Naor 1991]

Four models of computing

SLOCAL
distributed,
sequential

LOCAL
distributed,
parallel

online LOCAL
centralized

dynamic LOCAL
centralized

SLOCAL
distributed,
sequential

LOCAL
distributed,
parallel

online
LOCAL
centralized

dynamic
LOCAL
centralized

LOCAL

distributed,
parallel

Each node **in parallel**:

- looks at its radius-T neighborhood
- picks its output based on this information

(nodes have unique identifiers)

SLOCAL
distributed,
sequential

LOCAL
distributed,
parallel

online
LOCAL
centralized

dynamic
LOCAL
centralized

SLOCAL

distributed,
sequential

LOCAL

distributed,
parallel

online LOCAL

centralized

dynamic LOCAL

centralized

SLOCAL

distributed,
sequential

Each node in a **sequential, adversarial order**:

- looks at its radius-T neighborhood
- picks its output & state based on this information

SLOCAL
distributed,
sequential

LOCAL
distributed,
parallel

online LOCAL
centralized

dynamic LOCAL
centralized

SLOCAL
distributed,
sequential

LOCAL
distributed,
parallel

online LOCAL
centralized

dynamic LOCAL
centralized

Graph **constructed** by an adversary that adds nodes and edges one by one

We can **see everything**

We can **change** our output only within distance *T* from a point of change

dynamic

LOCAL

centralized

SLOCAL
distributed,
sequential

LOCAL
distributed,
parallel

online
LOCAL
centralized

dynamic
LOCAL
centralized

SLOCAL
distributed,
sequential

LOCAL
distributed,
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online LOCAL
centralized

dynamic LOCAL
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Some unknown input graph is **revealed** piece by piece:

- adversary points at a node v
- we can see the radius-T neighborhood of v
- we have to choose the label for v

We can **remember** everything

online LOCAL

centralized

SLOCAL
distributed,
sequential

LOCAL
distributed,
parallel

online LOCAL
centralized

dynamic LOCAL
centralized

Genuinely different models

SLOCAL
distributed,
sequential

LOCAL
distributed,
parallel

online LOCAL
centralized

dynamic LOCAL
centralized

coloring

SLOCAL

distributed,
sequential

LOCAL

distributed,
parallel

online LOCAL

centralized

dynamic LOCAL

centralized

coloring

SLOCAL

distributed,
sequential

LOCAL

distributed,
parallel

online LOCAL

centralized

cycle detection

dynamic LOCAL

centralized

coloring

SLOCAL

distributed, sequential

LOCAL

distributed, parallel

online LOCAL

centralized

cycle detection

dynamic LOCAL

centralized

leader election

Closely related models

SLOCAL
distributed,
sequential

LOCAL
distributed,
parallel

online LOCAL
centralized

dynamic LOCAL
centralized

works directly
(just ignore
local states)

SLOCAL
distributed,
sequential

LOCAL
distributed,
parallel

online LOCAL
centralized

dynamic LOCAL
centralized

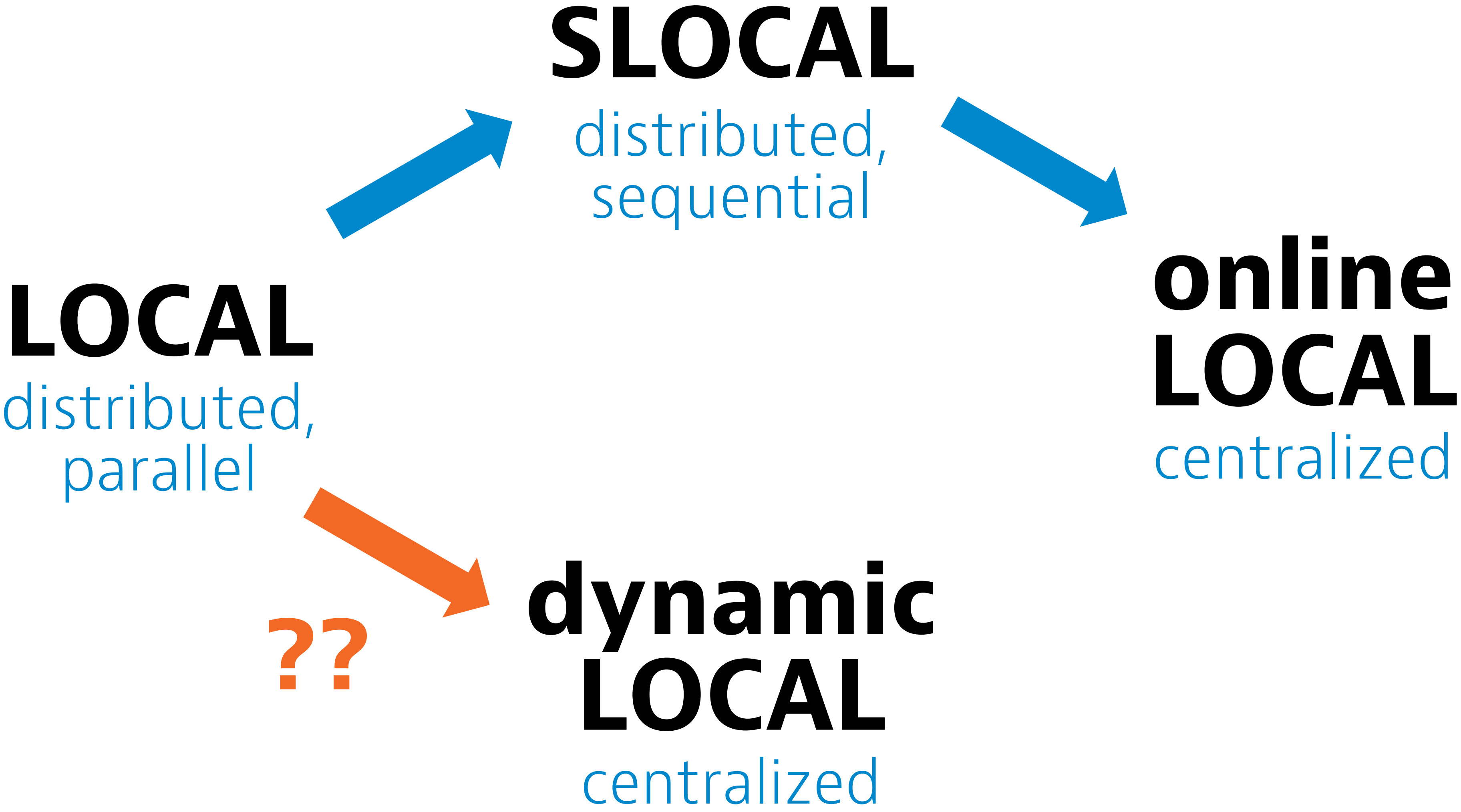
works directly
(just ignore
global view)

SLOCAL
distributed,
sequential

LOCAL
distributed,
parallel

**online
LOCAL**
centralized

**dynamic
LOCAL**
centralized


SLOCAL
distributed,
sequential
LOCAL
distributed,
parallel
online
LOCAL
centralized
dynamic
LOCAL
centralized
??

SLOCAL
distributed,
sequential

LOCAL
distributed,
parallel

online LOCAL
centralized

dynamic LOCAL
centralized

just recompute
everything after
each change!

Equivalent perspectives

- **Output of node *v* only depends on inputs of nodes *u* with dist(*u*, *v*) ≤ *T***
  - this is what we have by definition in LOCAL algorithms

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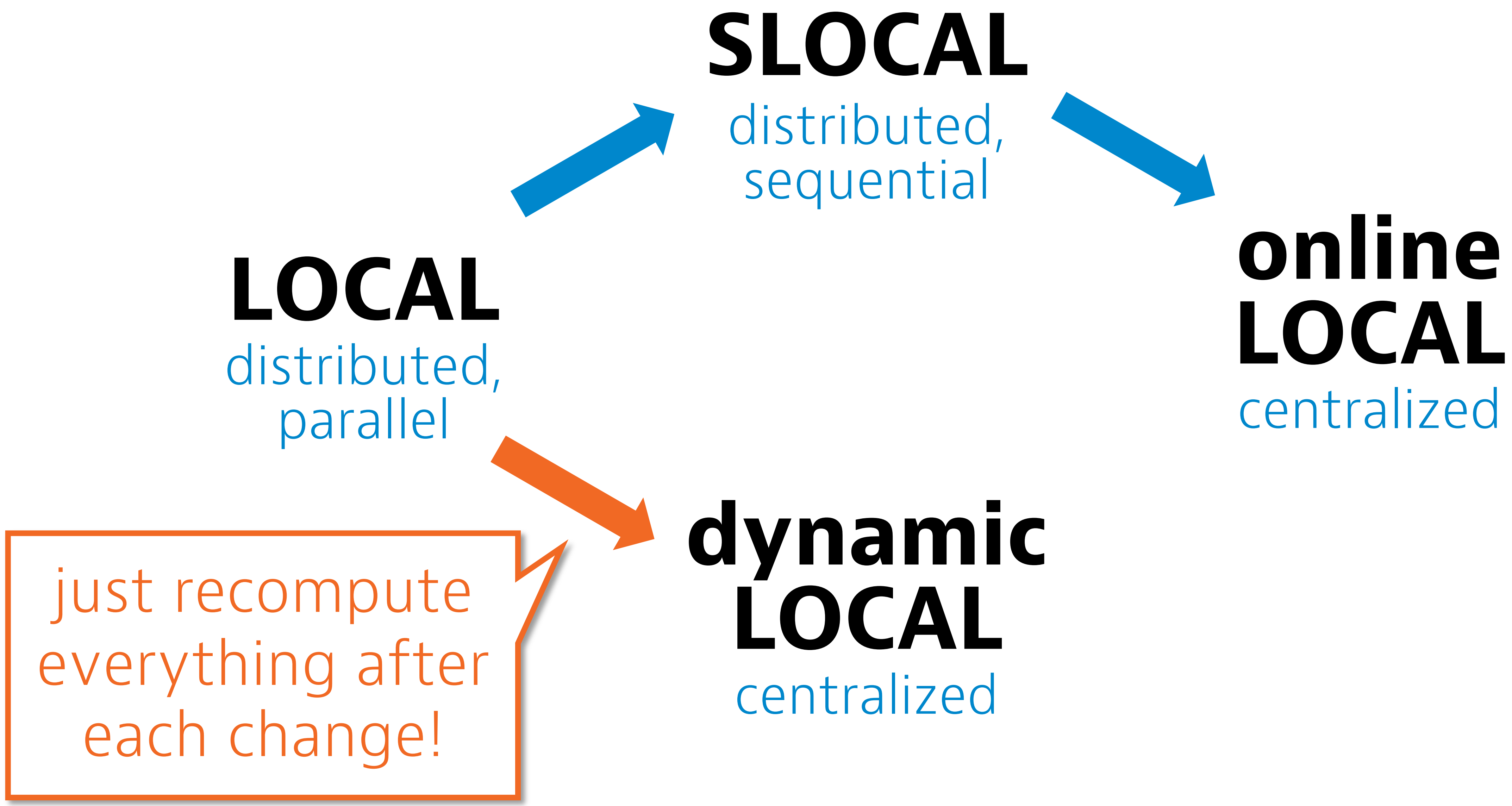
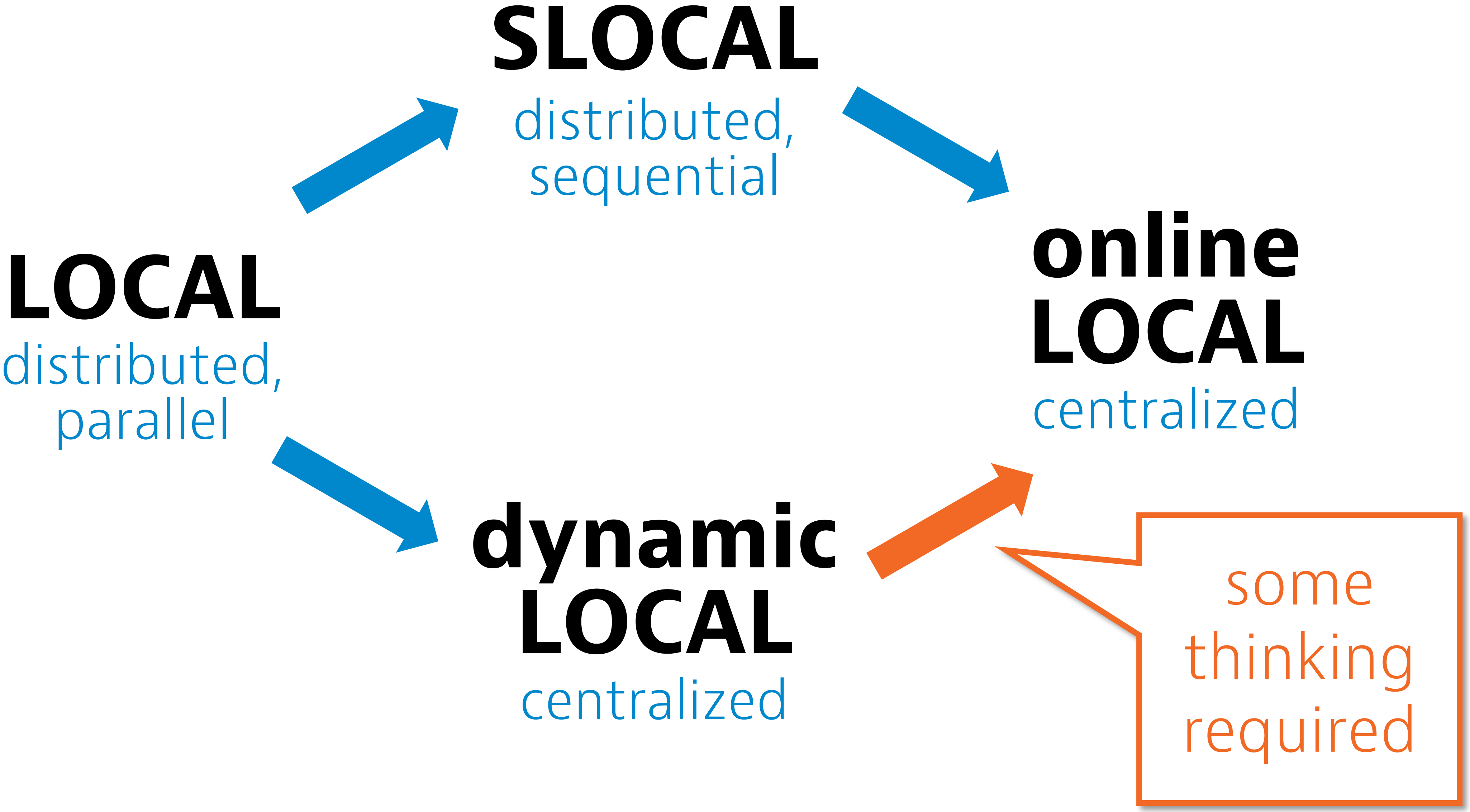
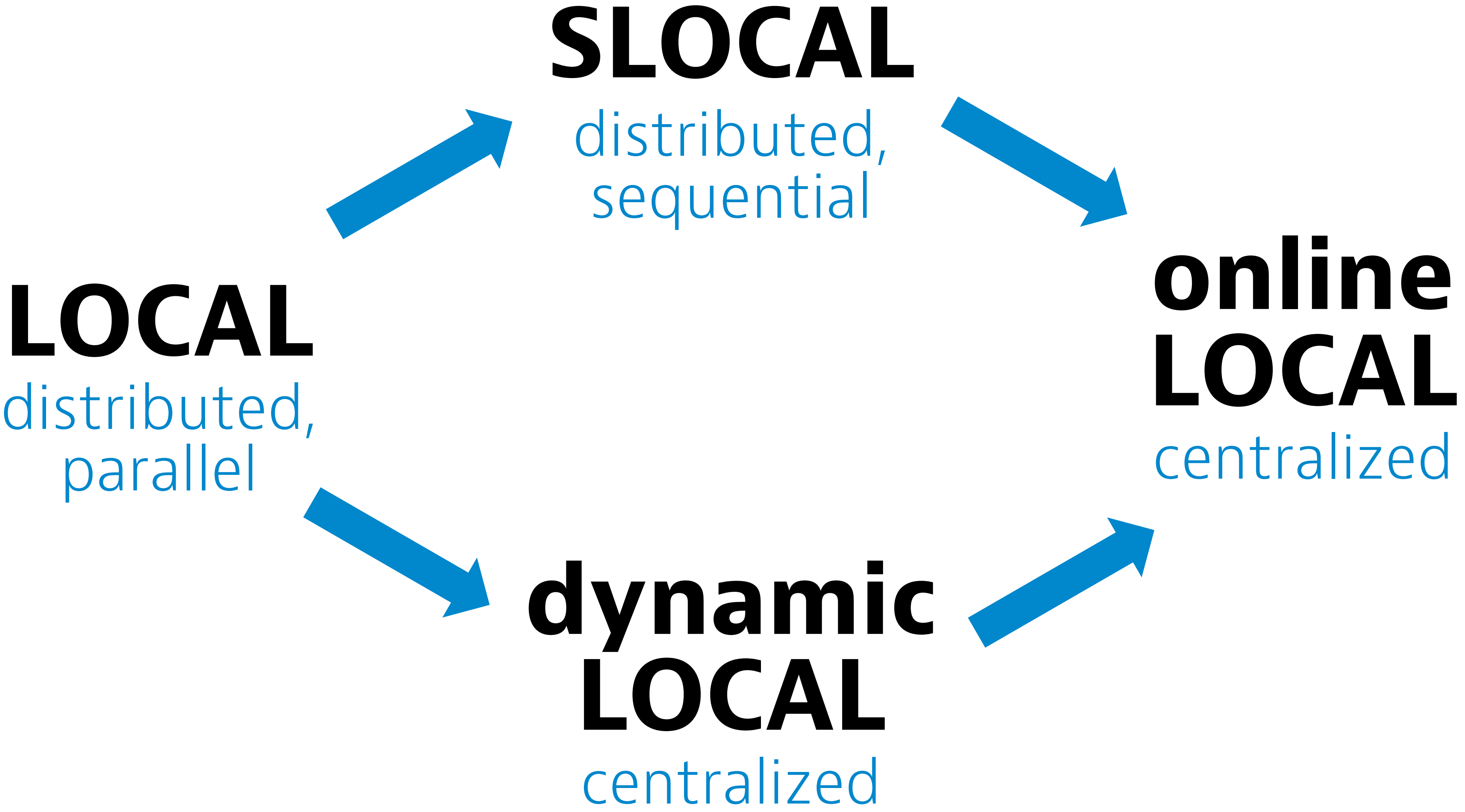
- **Output of node *v* only depends on inputs of nodes *u* with dist(*u*, *v*) ≤ *T***
  - this is what we have by definition in LOCAL algorithms
- **Changes at node *u* can only influence outputs of nodes *v* with dist(*u*, *v*) ≤ *T***
  - this is enough to have a dynamic LOCAL algorithm


SLOCAL
distributed, sequential
LOCAL
distributed, parallel
online LOCAL
centralized
dynamic LOCAL
centralized
just recompute everything after each change!


SLOCAL
distributed,
sequential
LOCAL
distributed,
parallel
online
LOCAL
centralized
dynamic
LOCAL
centralized
some
thinking
required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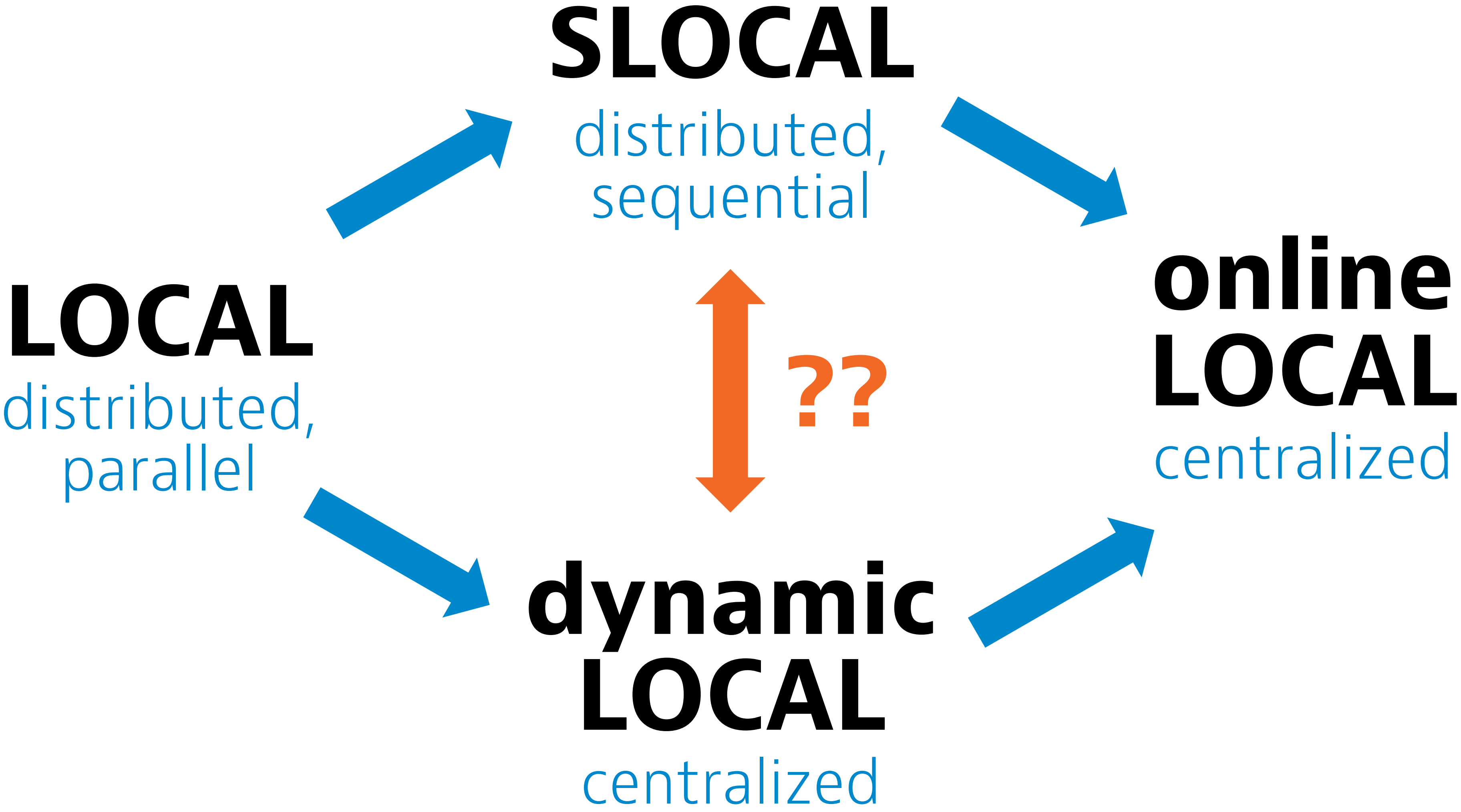
SLOCAL
distributed,
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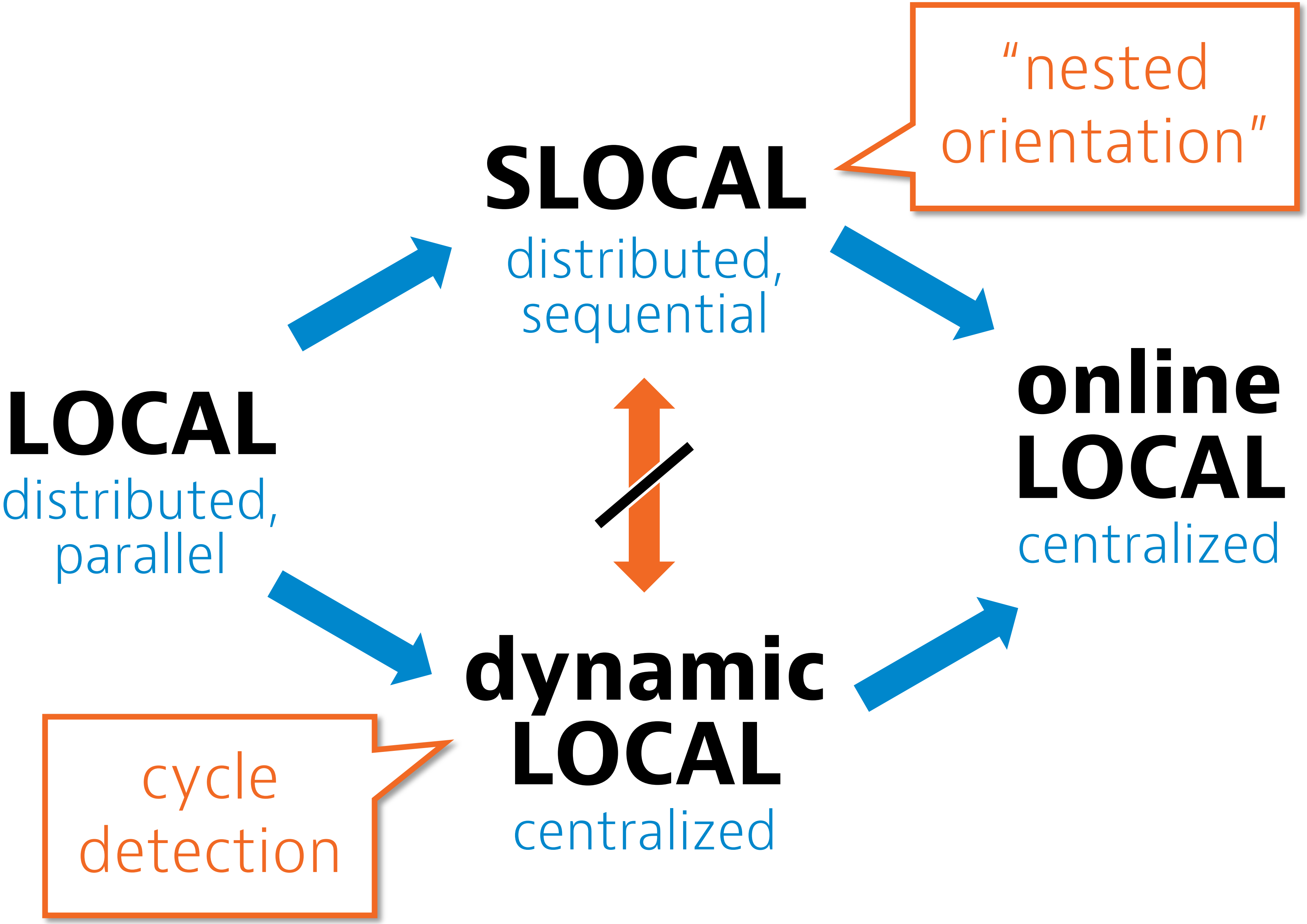
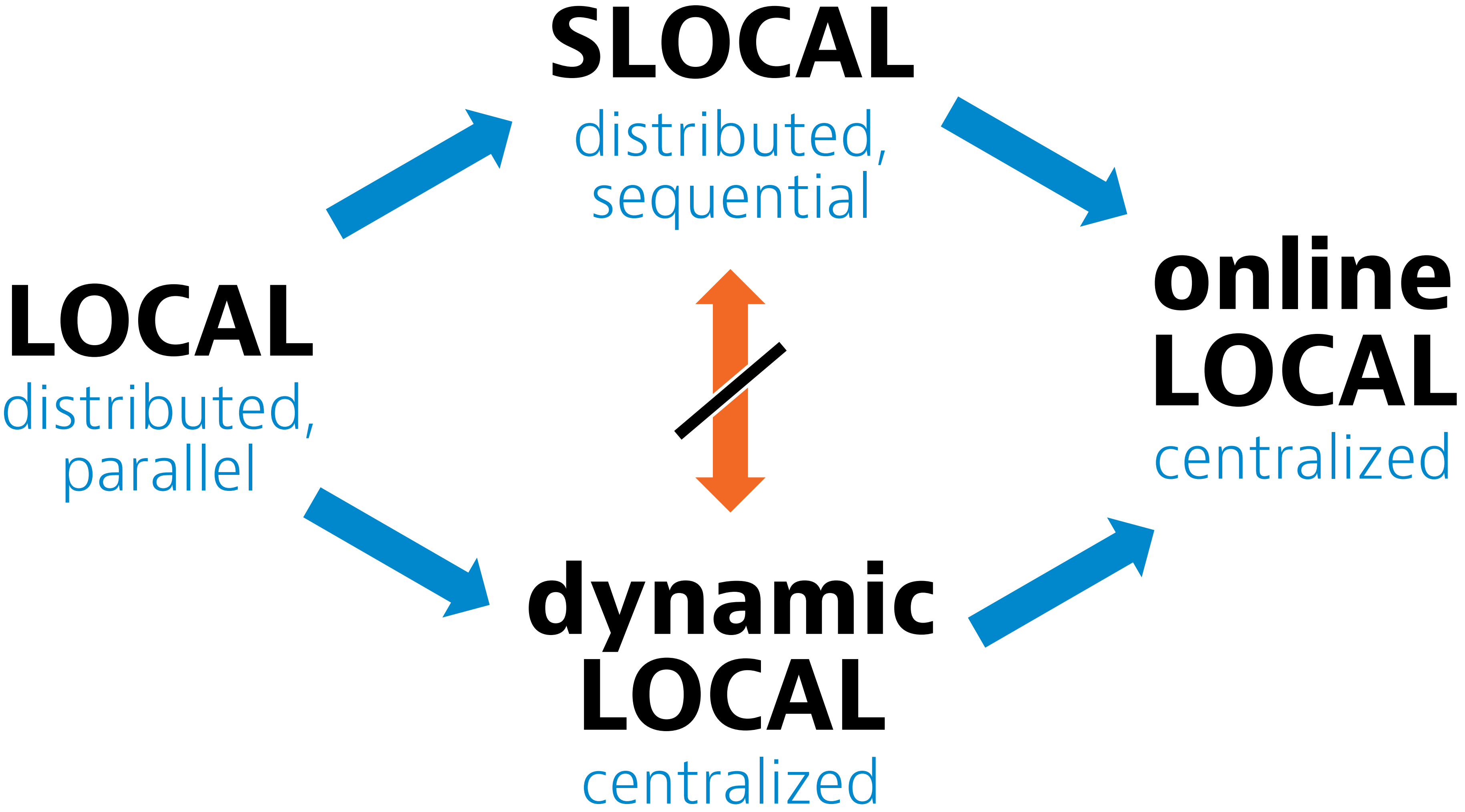


"nested orientation"
SLOCAL
distributed, sequential
LOCAL
distributed, parallel
online LOCAL
centralized
dynamic LOCAL
centralized
cycle detection


SLOCAL
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LOCAL
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dynamic
LOCAL
centralized

Collapse in rooted trees

LCLs in rooted trees

- **Rooted regular trees**
- **Locally checkable labelings** (LCLs)
  - solution valid if it "looks good everywhere"
  - example: 3-coloring
- ***In this setting all models equally strong!***

LCLs in rooted trees

SLOCAL

$\approx$ $\approx$

LOCAL

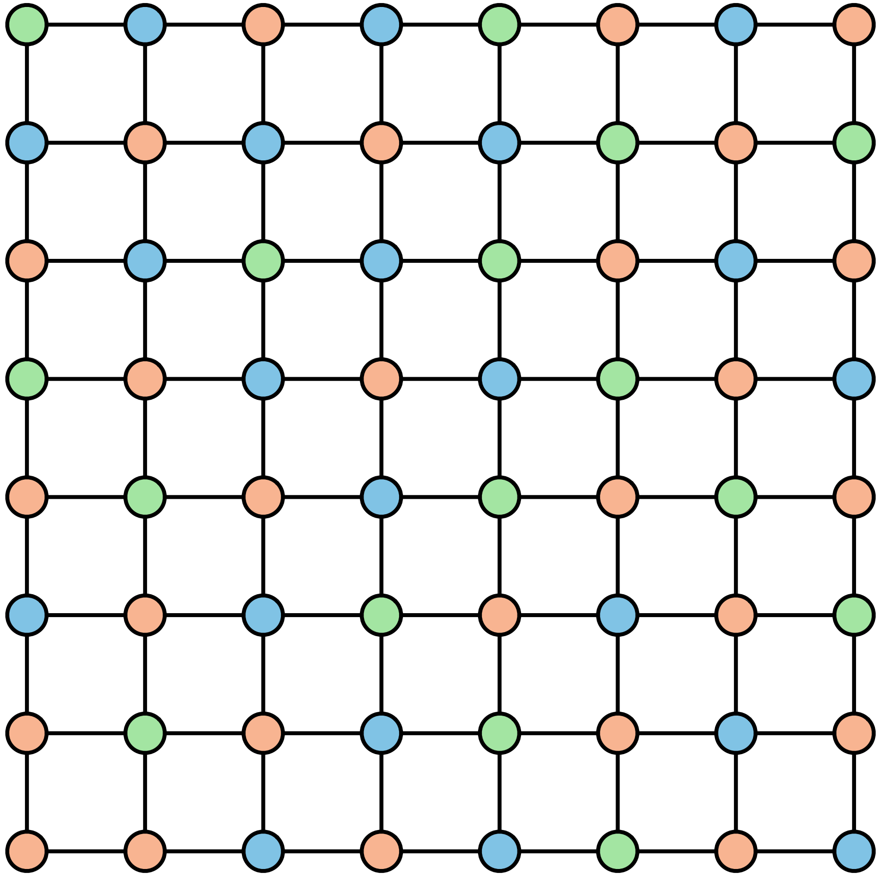
online LOCAL

$\approx$ $\approx$

dynamic LOCAL

Case study: grids

Coloring grids



Coloring grids

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  - **online-LOCAL: $O(\log n)$**

Coloring grids

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  - online-LOCAL: $O(\log n)$ — **is this tight?**

Coloring grids

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- **4-coloring:** local in all models (hard to see)
- **3-coloring:**
  - LOCAL, SLOCAL: global
  - online-LOCAL: $O(\log n)$ — **is this tight?**
  - dynamic-LOCAL: **open**

Distinct in general, equivalent for LCLs in trees