Brief Announcement: **Linial's Lower Bound Made Easy**

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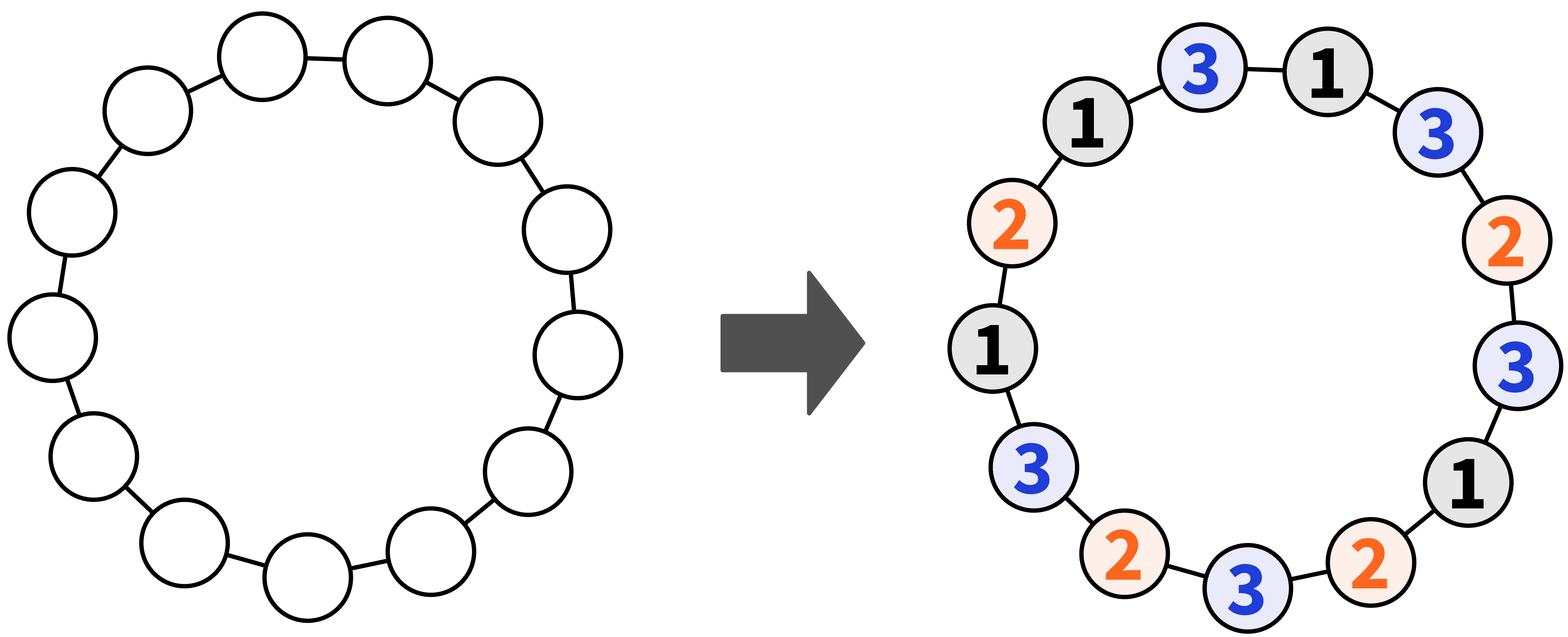
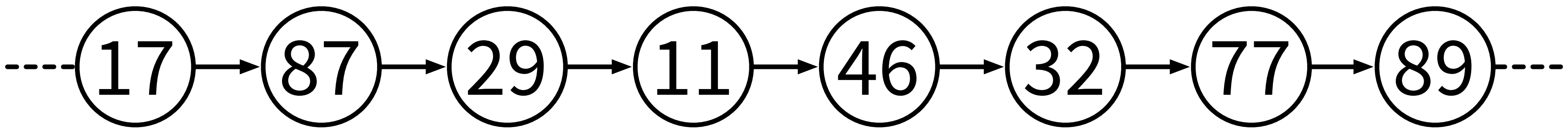
Problem:

3-colouring cycles



Problem:
3-colouring cycles

- **Directed cycle with *n* nodes**
- ***O*(log *n*)-bit identifiers, LOCAL model**

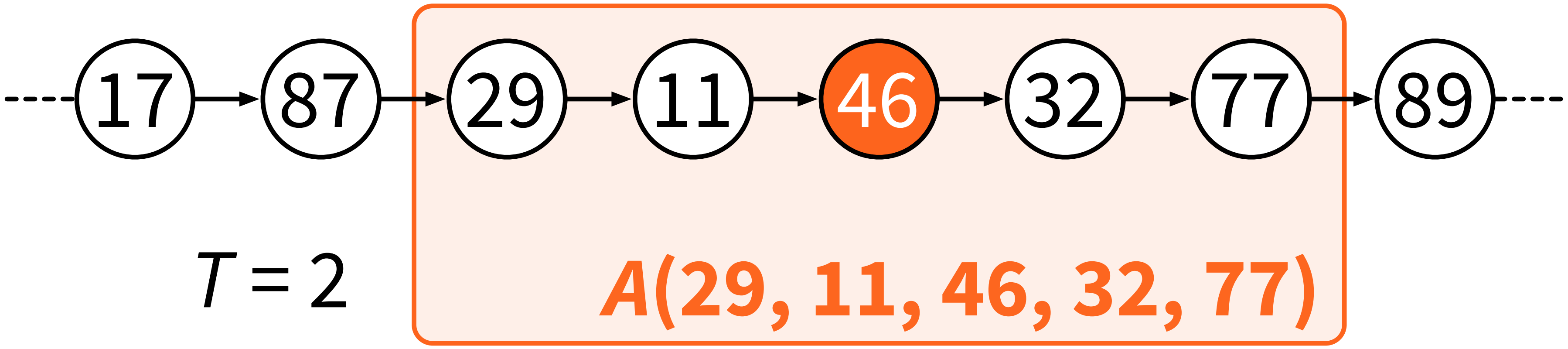


Problem:
3-colouring cycles

- **Linial (1992): requires at least**
  ½ $\log^*(n) - 1$ communication rounds
- **Today: same result, simpler proof**
  - **student-friendly**, self-contained, **< 2 pages**
  - no references to neighbourhood graphs, line graphs, chromatic numbers …

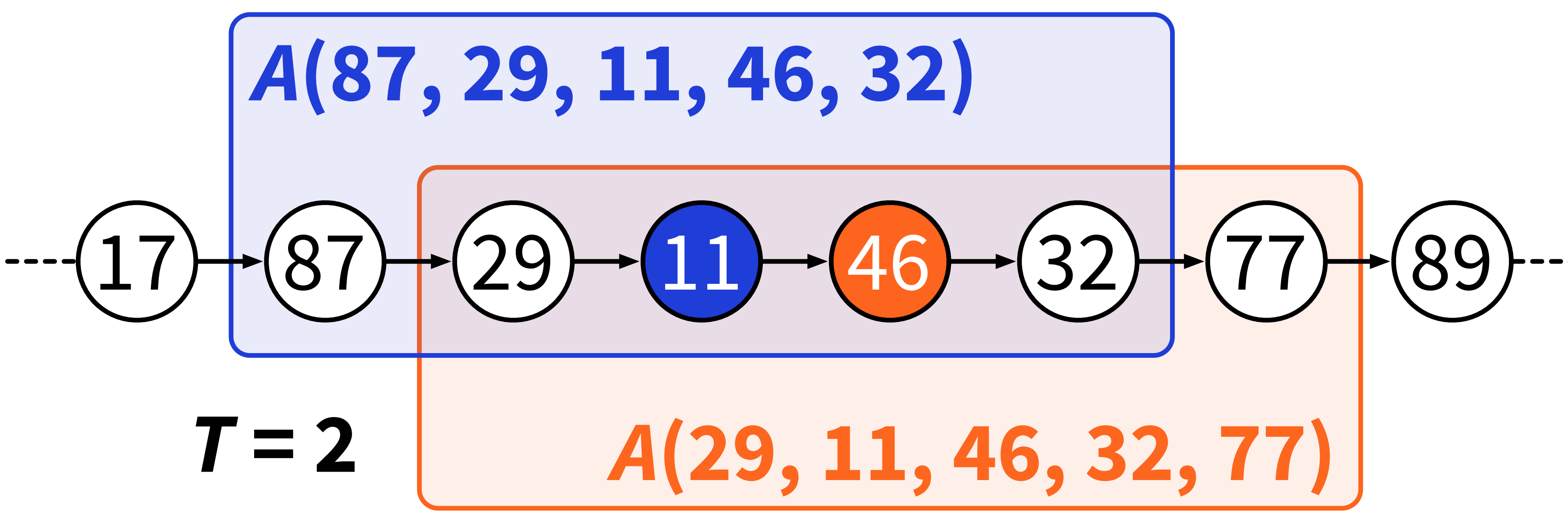
Colouring cycles in time T

- **Each node must output its own colour**
- **Running time T = output only depends on radius-T neighbourhood of the node**



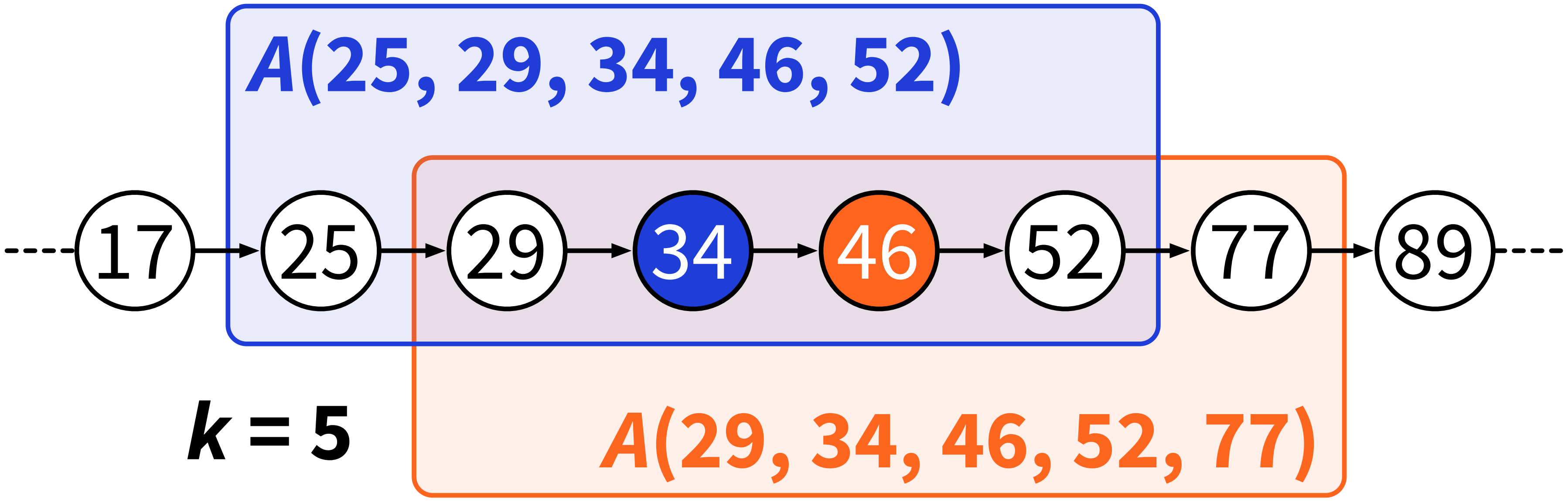
Colouring cycles in time T

$A(87, 29, 11, 46, 32) \neq A(29, 11, 46, 32, 77)$



k-ary *c*-colouring function

$A(25, 29, 34, 46, 52) \neq A(29, 34, 46, 52, 77)$



k-ary c-colouring function

- $A(x_1, x_2, \ldots, x_k) \in \{1, 2, \ldots, c\}$
  for all $1 \leq x_1 < x_2 < \ldots < x_k \leq n$

- $A(x_1, x_2, \ldots, x_k) \neq A(x_2, x_3, \ldots, x_{k+1})$
  for all $1 \leq x_1 < x_2 < \ldots < x_{k+1} \leq n$

k-ary c-colouring function

- **Assume:** A is a distributed algorithm that finds a 3-colouring in directed n-cycles in time T
- **Then:** A is a k-ary 3-colouring function for $k = 2T + 1$
- **Plan:** show that $k + 1 \geq \log^* n$

Lemma 1

- **If there is a 1-ary c-colouring function, then $c \geq n$**
- **Proof:**
  - pigeonhole principle

Lemma 2

- **Given: a *k*-ary *c*-colouring function *A***
- **We can construct:**
  a $(k-1)$-ary 2^c-colouring function *B*
- **Proof:**
  - $B(x_1, x_2, \ldots, x_{k-1}) = \{A(x_1, x_2, \ldots, x_{k-1}, y) : y > x_{k-1}\}$

Iterate Lemma 2

$^{i}2 = 2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}$ (i twos)

- **k-ary 3-colouring function →**
  k-ary $^{2}2$-colouring function →
  ($k - 1$)-ary $^{3}2$-colouring function →
  ($k - 2$)-ary $^{4}2$-colouring function →
  …
  1-ary $^{k+1}2$-colouring function

Conclusion

$^{i}2 = 2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}$ (i twos)

- **Lemma 2:**
  - k-ary **3**-colouring function →
    1-ary $^{k+1}$**2**-colouring function
- **Lemma 1:**
  - $^{k+1}2 \geq n$ (that is, $k + 1 \geq \log^* n$)