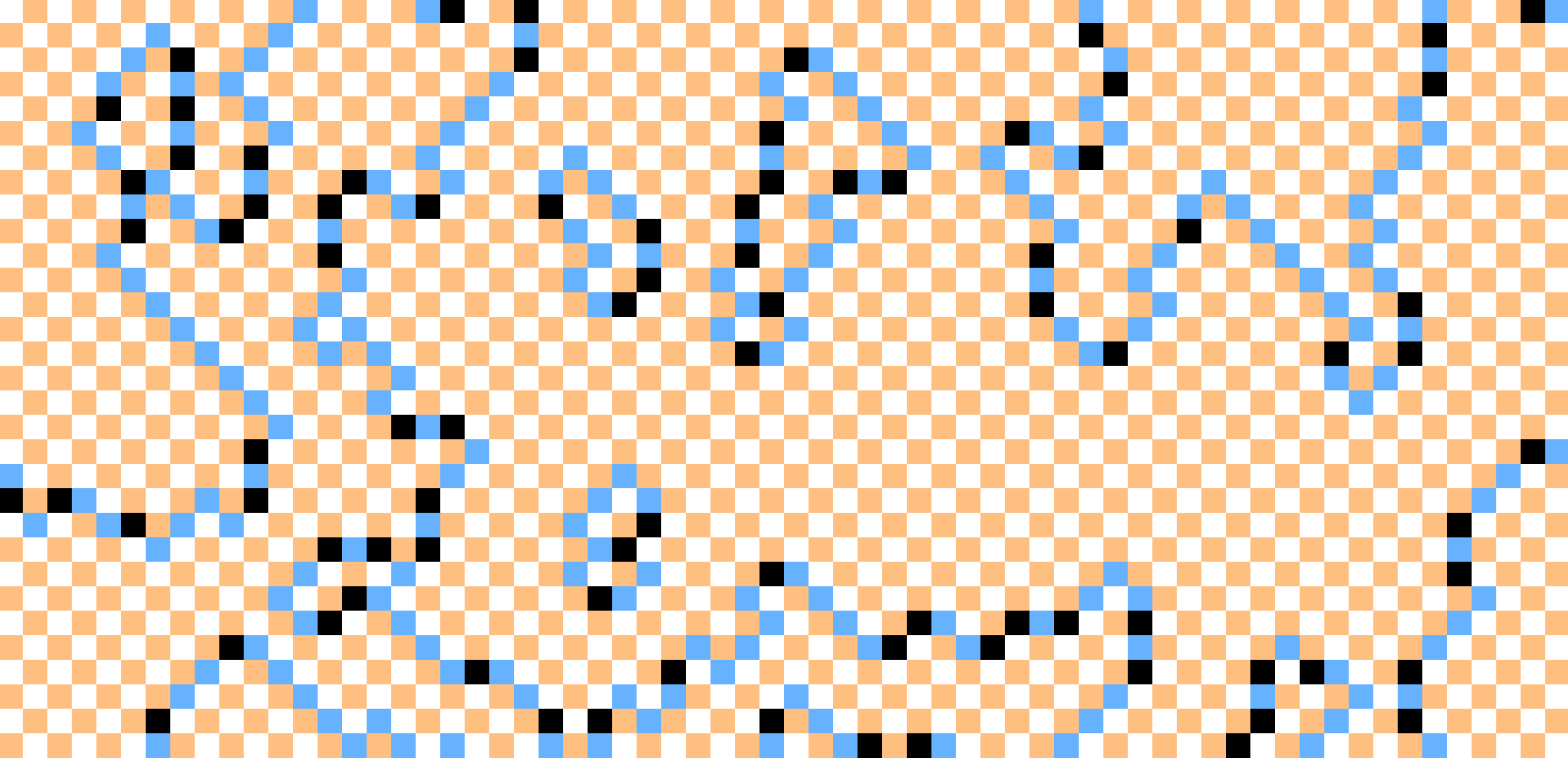

LCL Problems on Grids

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joint work with:

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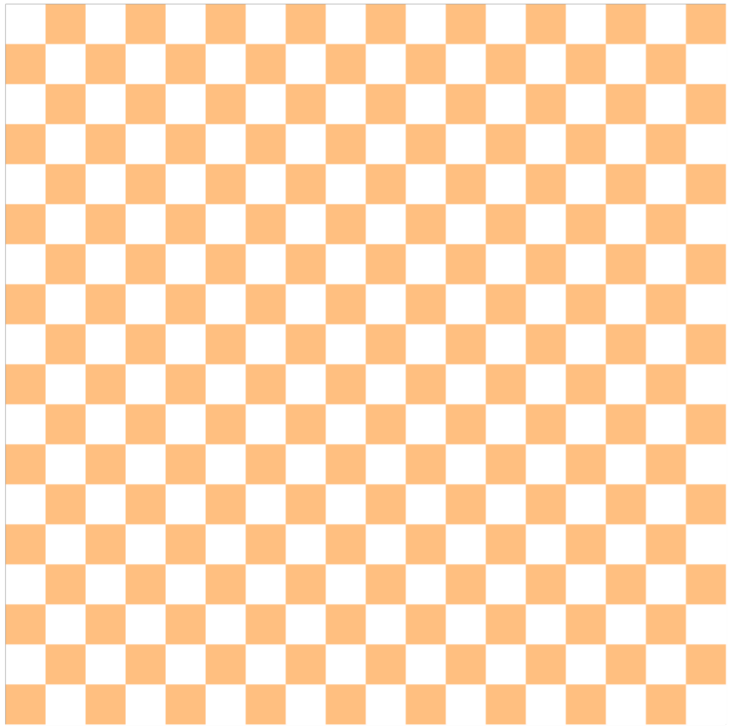
- Sebastian Brandt, Przemysław Uznański **(ETH Zürich)**
- Juho Hirvonen **(Paris Diderot)**
- Joel Rybicki **(U. Helsinki)**

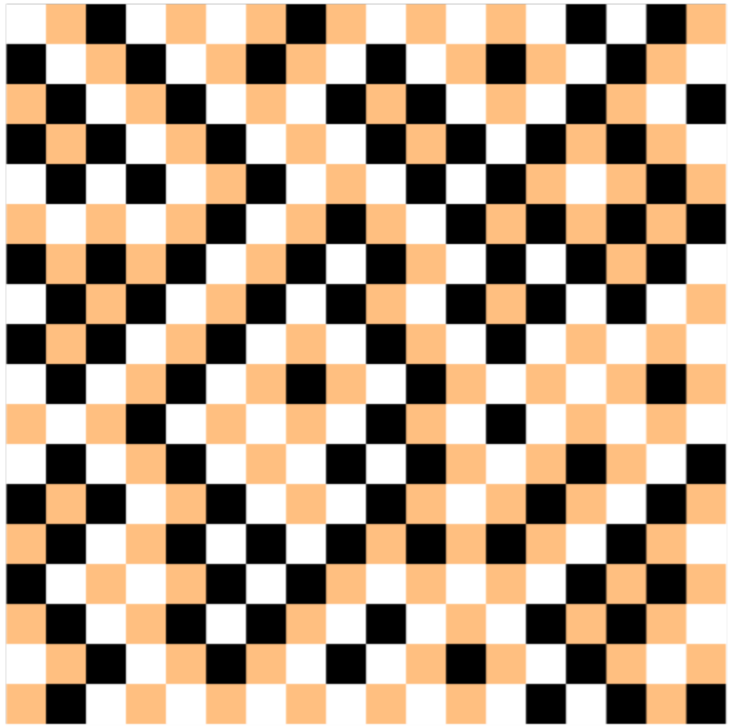
(arXiv:1702.05456)

Setting: LOCAL model, 2D grids

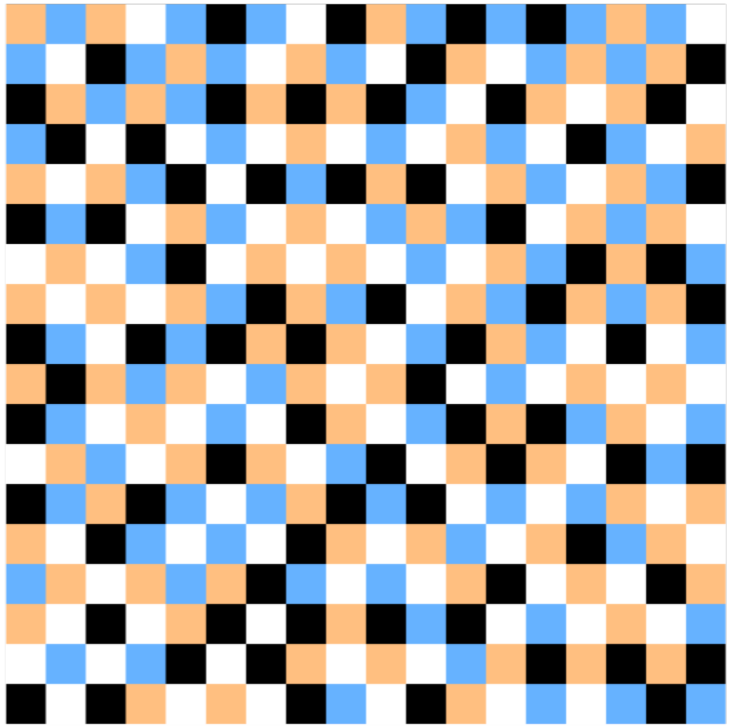
2-colouring



3-colouring



4-colouring



1.

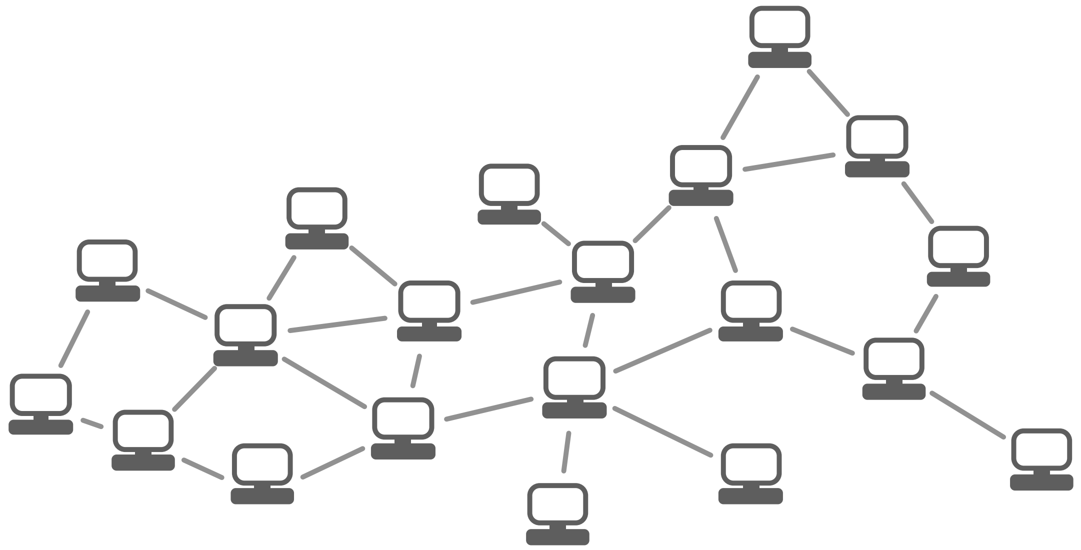
Introduction

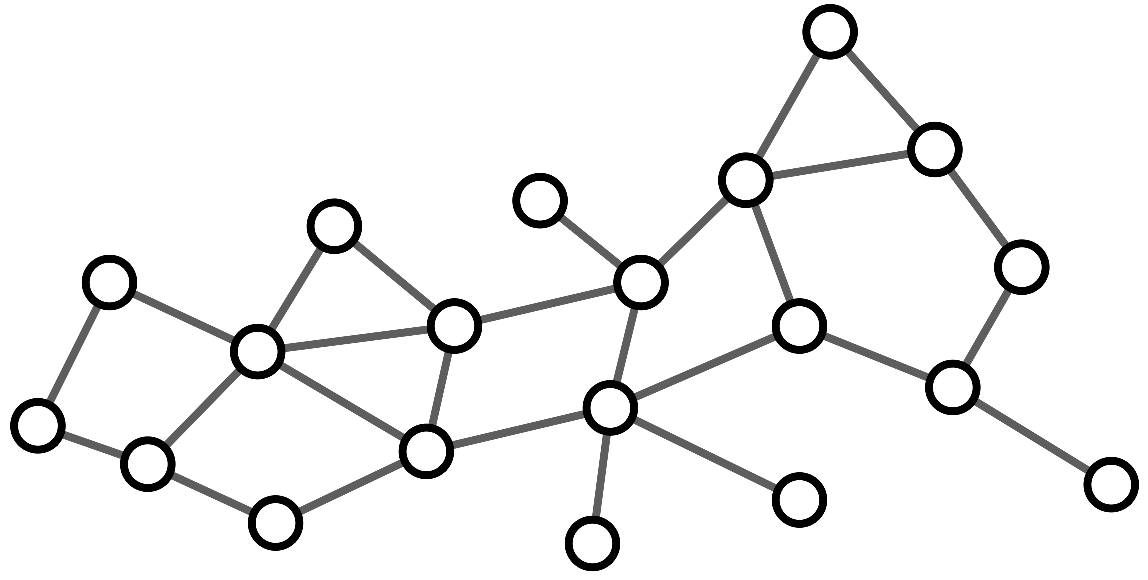
Setting:

LOCAL model

- graph = computing network
- graph = input instance

- **input:** local information at each node
- **output:** local structure of solution



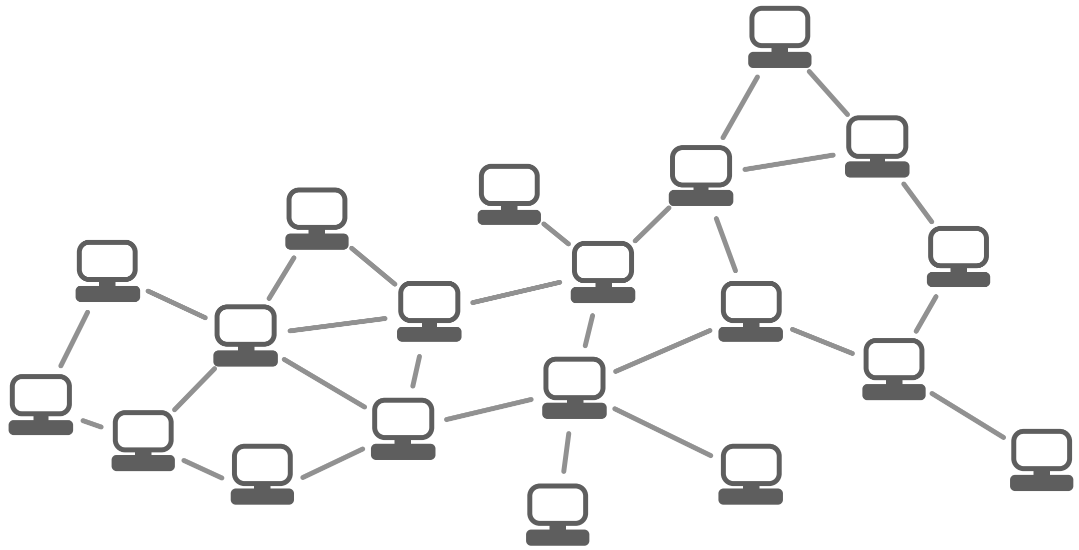


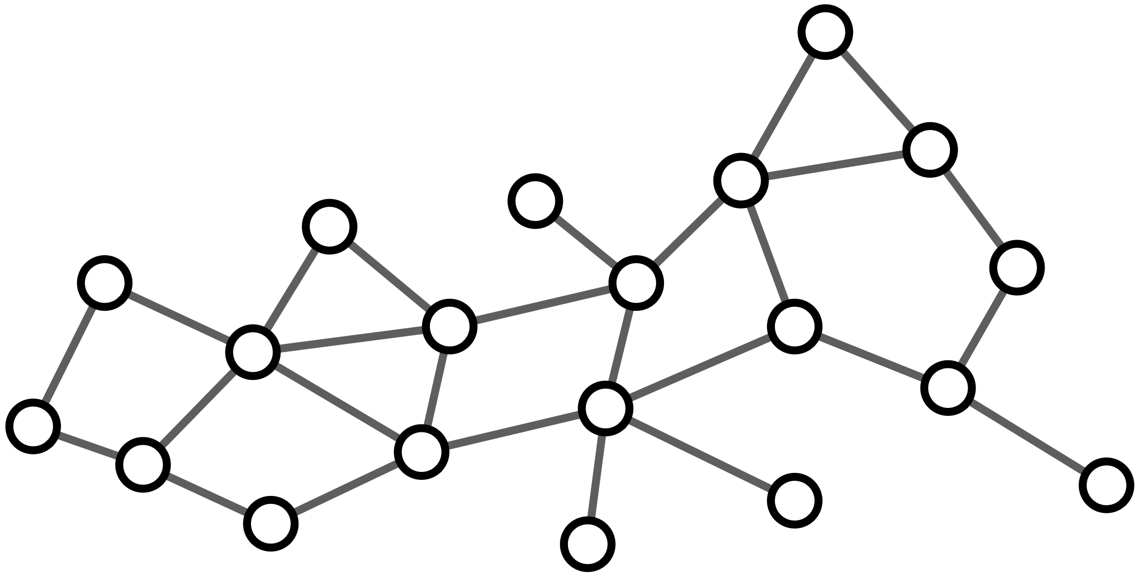
Setting:

LOCAL model

- **LOCAL model**
    - unique identifiers
    - synchronous communication rounds
    - time measure: ***number of rounds***
    - unlimited local computation, unlimited message size

- **This work:** deterministic algorithms





Setting:

LCL problems

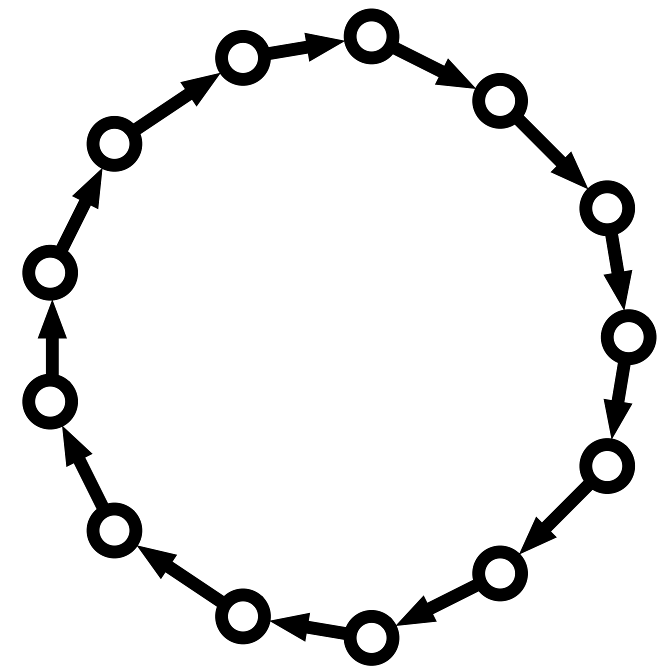
- **LCL = locally checkable labelling**
  - Naor and Stockmeyer (1995)

  - constant-size labels
  - validity of a solution checkable in **$O(1)$**-radius neighbourhood of each node

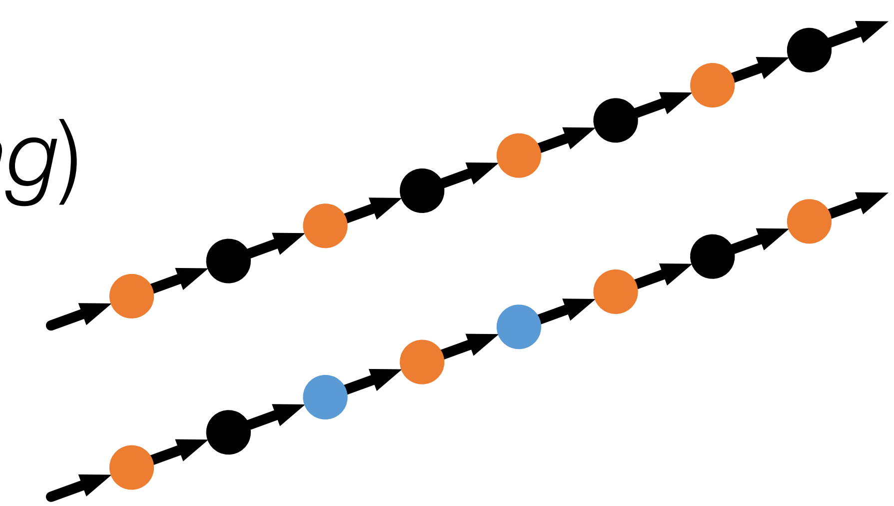
  - *maximal independent set*, *maximal matching*, *vertex colouring*, *edge colouring* ...

Background:

LCLs on cycles

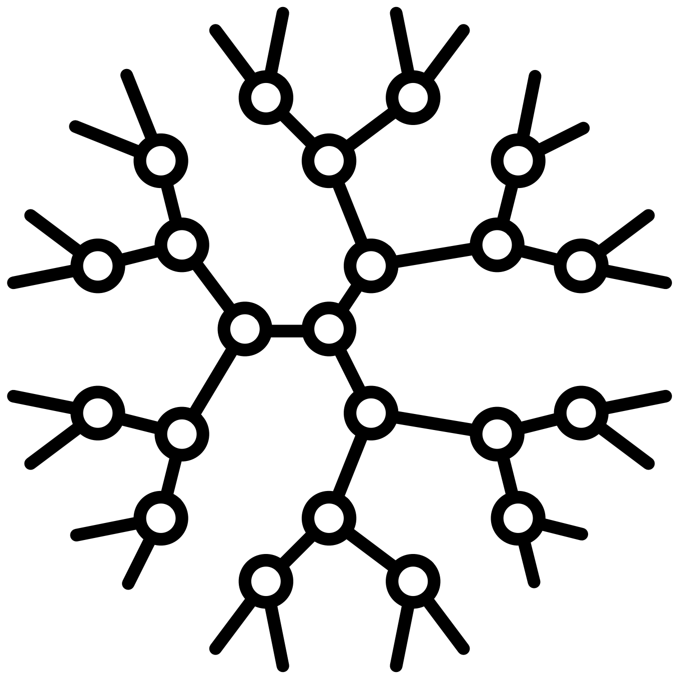


- **Directed cycles**
  - Cole and Vishkin (1986), Linial (1992)…
- **Well-understood classification**
  - **$\Theta(1)$** time: “**trivial**”
  - **$\Theta(\log^* n)$** time: “**local**” (3-*colouring*)
  - **$\Theta(n)$** time: “**global**” (2-*colouring*)
  - classification **decidable**



Background:

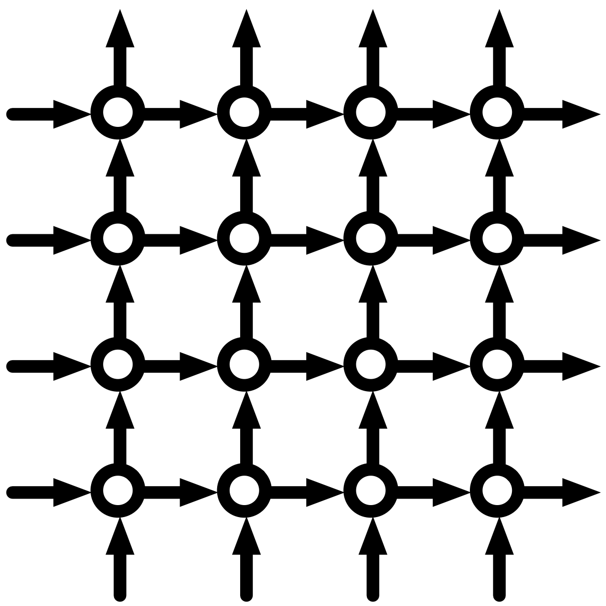
LCLs on general graphs



- **General (bounded-degree) graphs**
  - lots of ongoing work
  - challenge: ***expander graphs***
- A more complicated landscape
  - **gaps:** nothing is **$\omega(\log^* n)$** and **$o(\log n)$**
  - **intermediate problems: $\Theta(\log n)$** complexity
  - Brandt et al. (2016), Chang et al. (2016), Ghaffari and Su (2017), …

This work:

2D grids



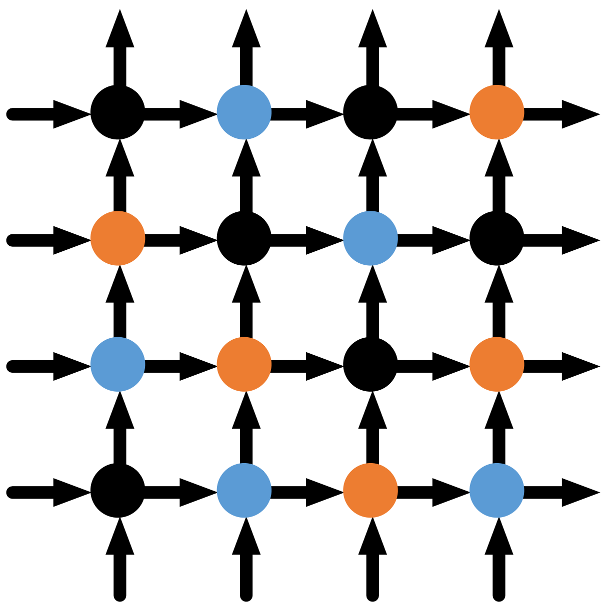
- **Oriented grids** (2D)
  - toroidal grid, $n \times n$ nodes, unique identifiers
  - consistent orientations north/east/south/west

- Generalisation of **directed cycles** (1D)

- Closer to real-world systems than expander-like worst-case constructions?

This work:

2D grids

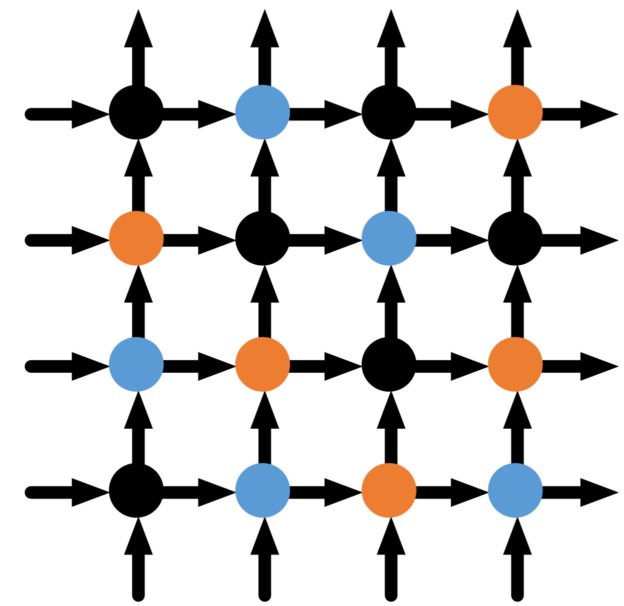


- **Vertex colouring** (deterministic)

- **2-colouring**: global, **$\Theta(n)$** rounds
- **3-colouring**: ???
- **4-colouring**: ???
- **5-colouring**: local, **$\Theta(\log^* n)$** rounds

This work:

2D grids



- **Vertex colouring** (deterministic)

- **2-colouring**: global, **$\Theta(n)$** rounds
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- **4-colouring**: local, **$\Theta(\log^* n)$** rounds
- **5-colouring**: local, **$\Theta(\log^* n)$** rounds

2.

Classification of LCL problems on grids

Main theorem:

Classification on grids

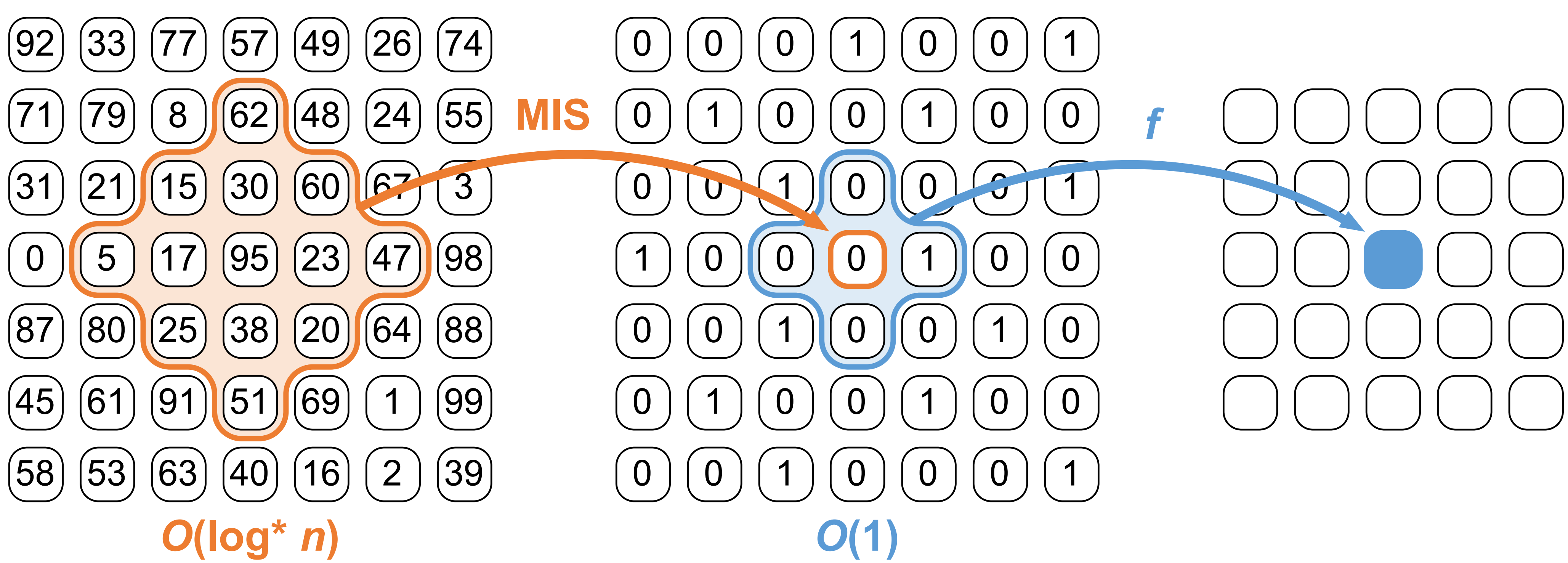
- LCL problems on 2D grids have ***exactly three possible deterministic complexities***:
    - $\Theta(1)$ time: “**trivial**”
    - $\Theta(\log^* n)$ time: “**local**”
    - $\Theta(n)$ time: “**global**”

- **Why?**
    - $o(\log^* n)$ time implies $O(1)$ time (Naor–Stockmeyer)
    - $o(n)$ time implies $O(\log^* n)$ time (this work)

Main theorem:

Normalisation/speed-up

- **Theorem:** Any deterministic ***o*(*n*)**-time algorithm can be translated to a "*normal form*":
  1. fixed **Θ(log* *n*)**-time symmetry breaking component
  2. problem-specific ***O*(1)**-time component



Main theorem:

Normalisation, proof ideas

- For **any problem** P of complexity $o(n)$, there are constants k and r and function f such that P can be solved as follows:
  - **input:** 2D grid G with unique identifiers
  - find a maximal independent set in G^k
  - discard unique identifiers
  - apply function f to each $r \times r$ neighbourhood

Main theorem:

Normalisation, proof ideas

- **Why does this work?**
  - $o(n)$ algorithm A cannot see the whole graph
  - symmetry breaking gives ***locally unique identifiers***
  - pretend that instance has constant size
    → A still has to produce valid output

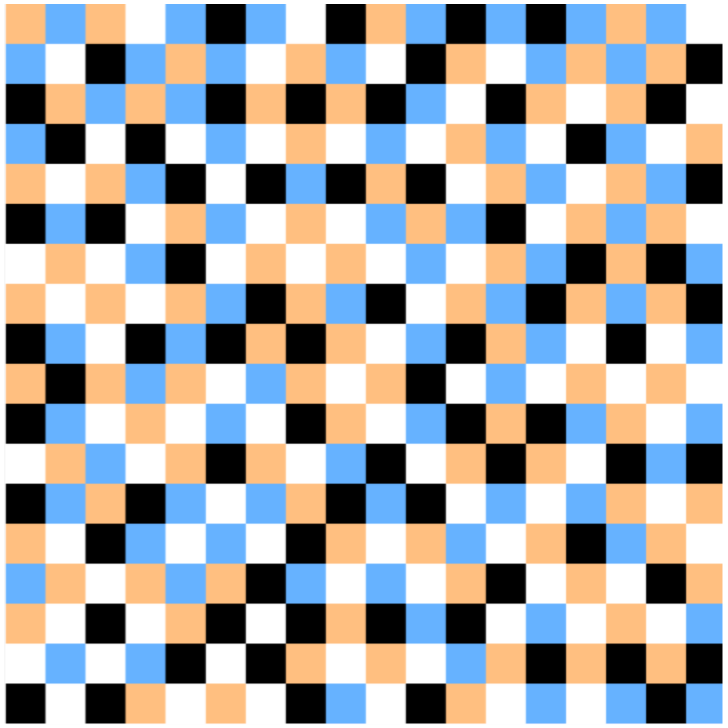
- Compare with speed-up for **general graphs**
  - $o(\log n) \to O(\log^* n)$
  - Chang et al. (2016)

3.

Vertex colouring
upper and lower bounds

Local problems:

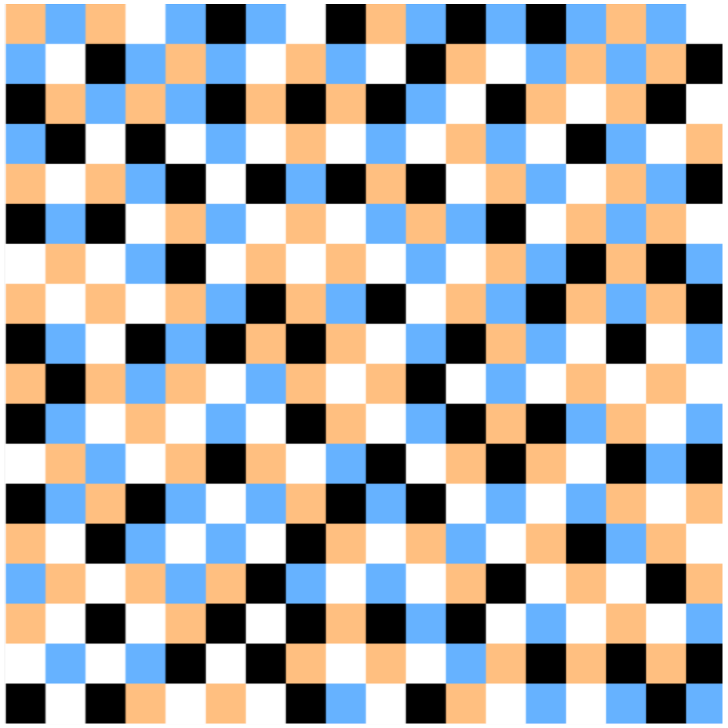
4-colouring



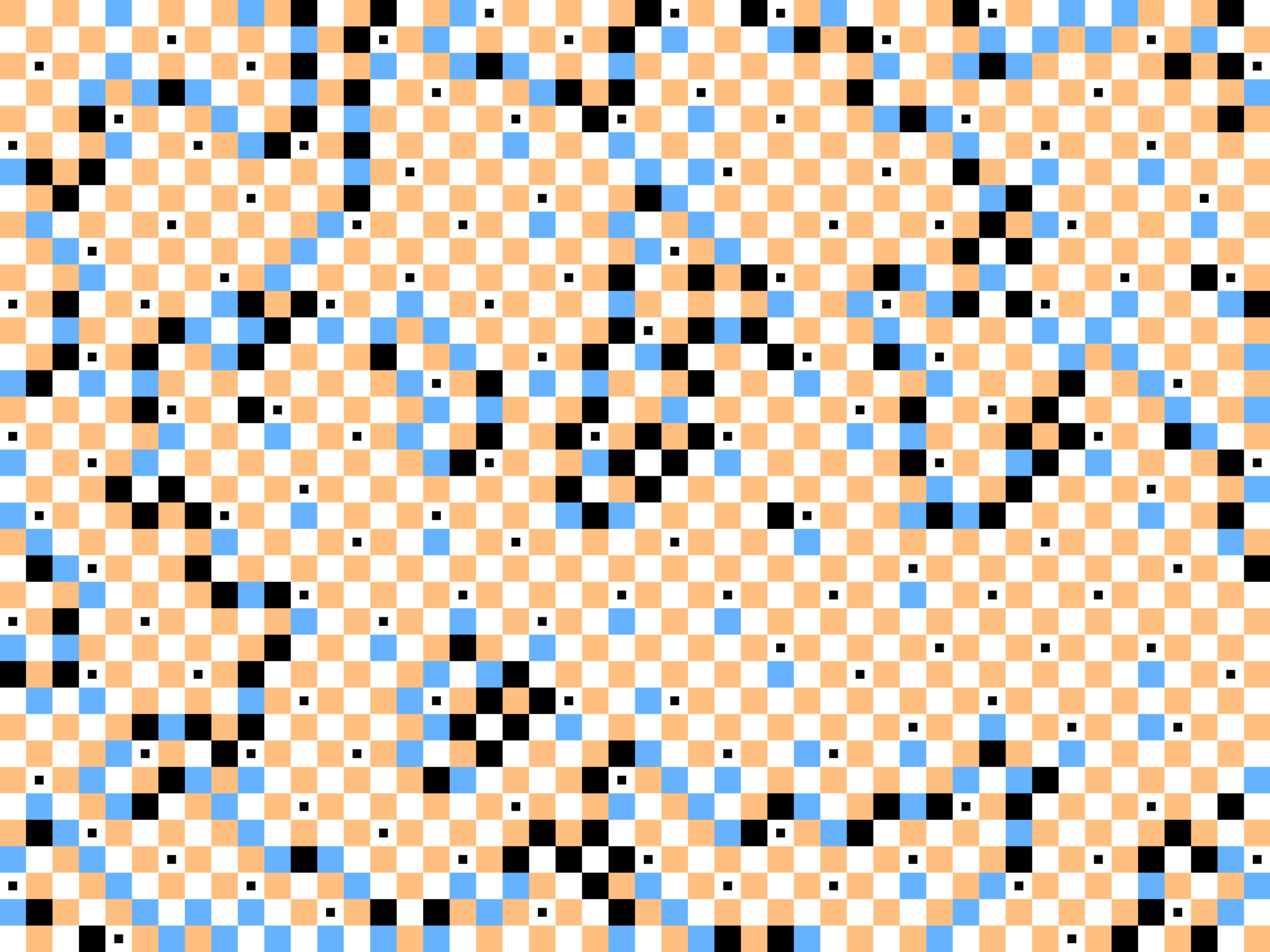
- **4-colouring** is ***local***
  - why?
- **First proof:** prior work and normalisation
  - **Δ-colouring** is **polylog(*n*)** for constant **Δ**
  - Panconesi and Srinivasan (1995)
  - normalisation → **O(log* *n*)**

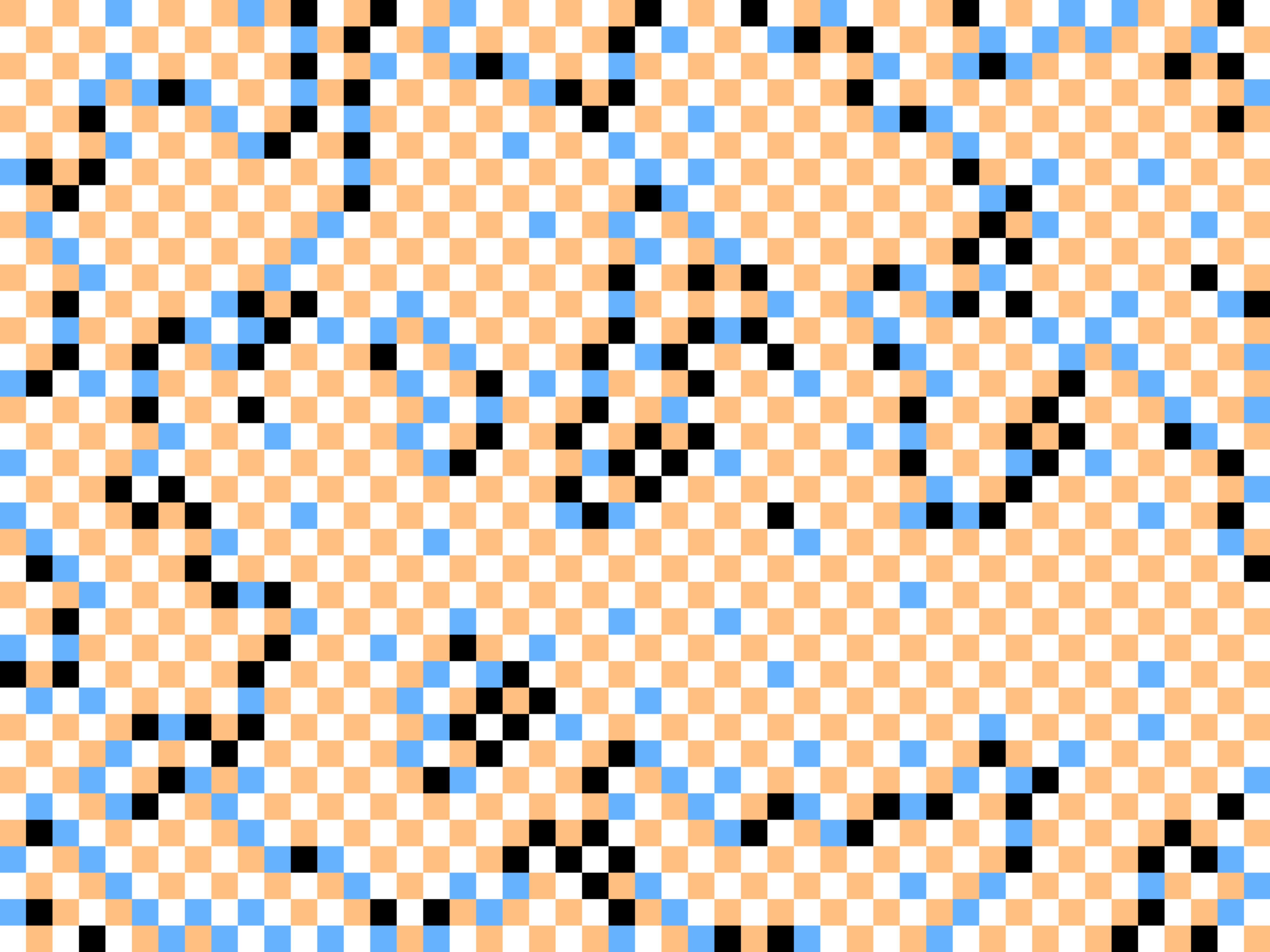
Local problems:

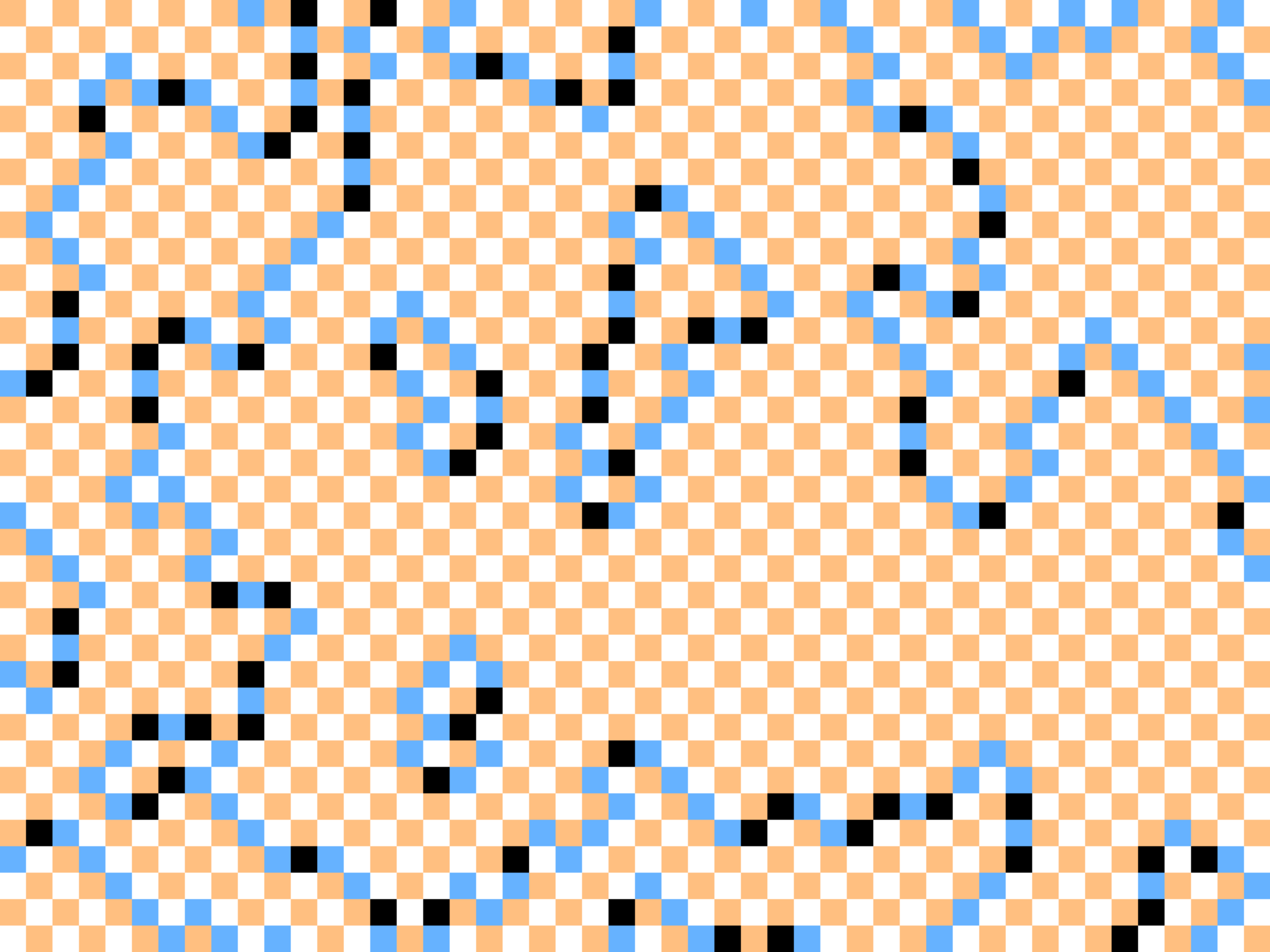
4-colouring



- **4-colouring** is ***local***
  - why?
- **Second proof: *synthesis***
  - guess it is local, use computers to find normal form
  - turns out it is enough to find an MIS in G^3, then consider 7×5 tiles
  - algorithm ≈ mapping $\{0, 1\}^{7 \times 5} \to \{1, 2, 3, 4\}$
  - only **2079** possible tiles, easy to find a solution







Local problems:

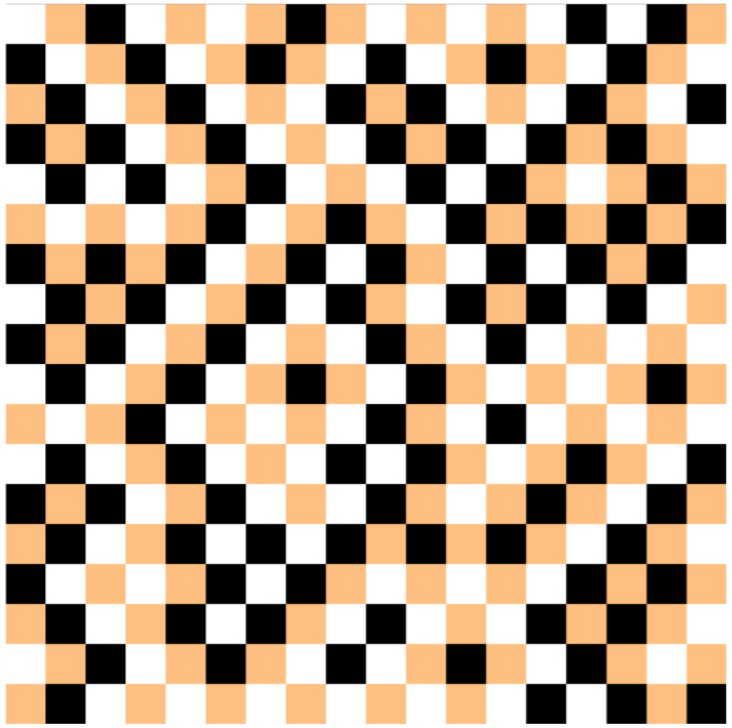
More on synthesis

- Can be applied to ***any* LCL *problem***
- However, classification on grids is **undecidable**
  - synthesis works if the problem is **local**
  - cannot give a negative answers for **global** problems

  - constants quite small in practice
  - more examples in the paper

Global problems:

2-colouring and 3-colouring



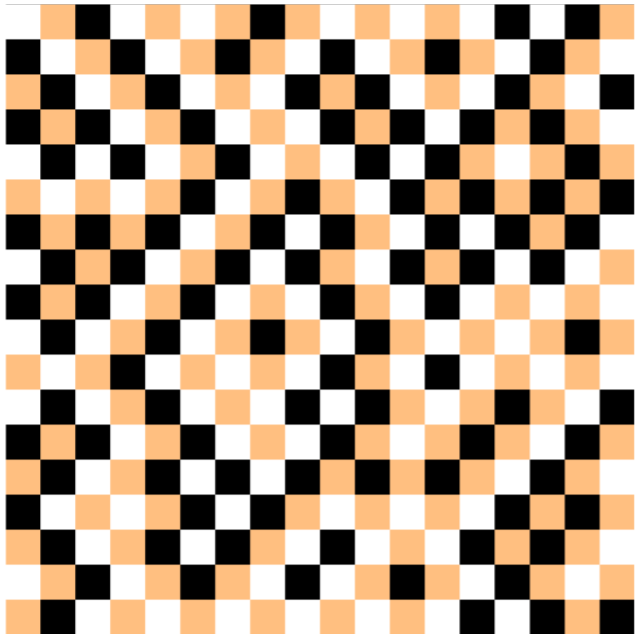
- **2-colouring** is ***global***
- **3-colouring** is ***global***
  - not local, but lower bound non-trivial
  - 3-colouring algorithm on grids solves "*sum coordination*" on **n-cycles**
  - "sum coordination" is global
  - reduction has ***topological*** flavour

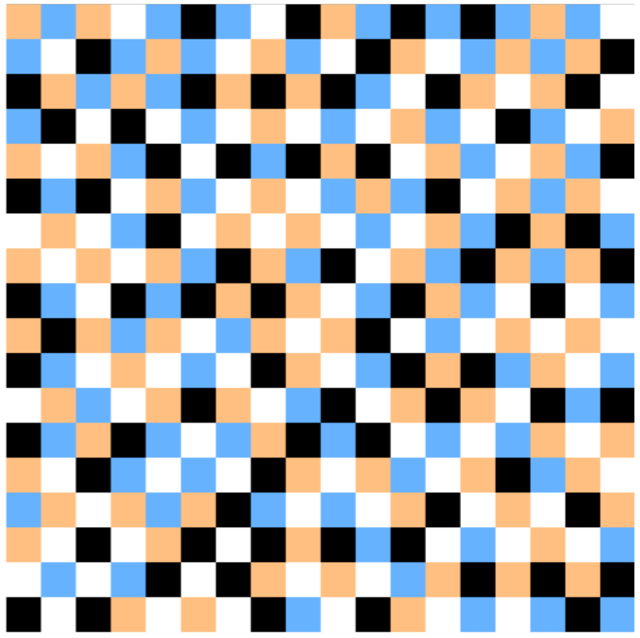
Vertex colouring:

Some remarks

- Human-designed **local** algorithm for 4-colouring on ***d-dimensional grids***

- Connection to ***finitary colourings***
    - **infinite** 2D grids
    - same proof techniques
    - upper and lower bounds
    - Holroyd et al. (2016)





4.

Conclusions

- **Generalisations**
  - **d-dimensional grids:** everything generalises
  - **bounded neighbourhood growth:** similar speed-up
  - randomised algorithms?

- **LCL landscape on general graphs still open**
  - $\Theta(n^{1/2})$ problems exist (this work)
  - $\Theta(n^{1/k})$ problems exist (Chang and Pettie 2017)
  - more **gap theorems**

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Thanks! Questions?