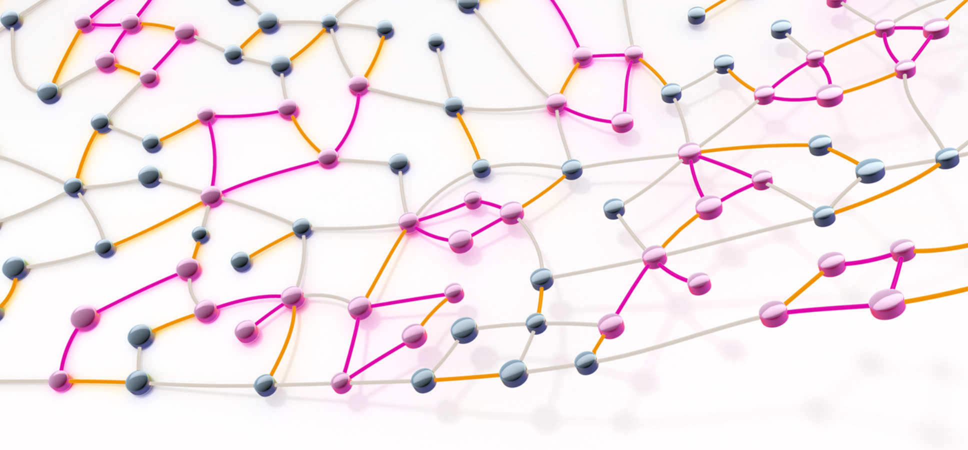


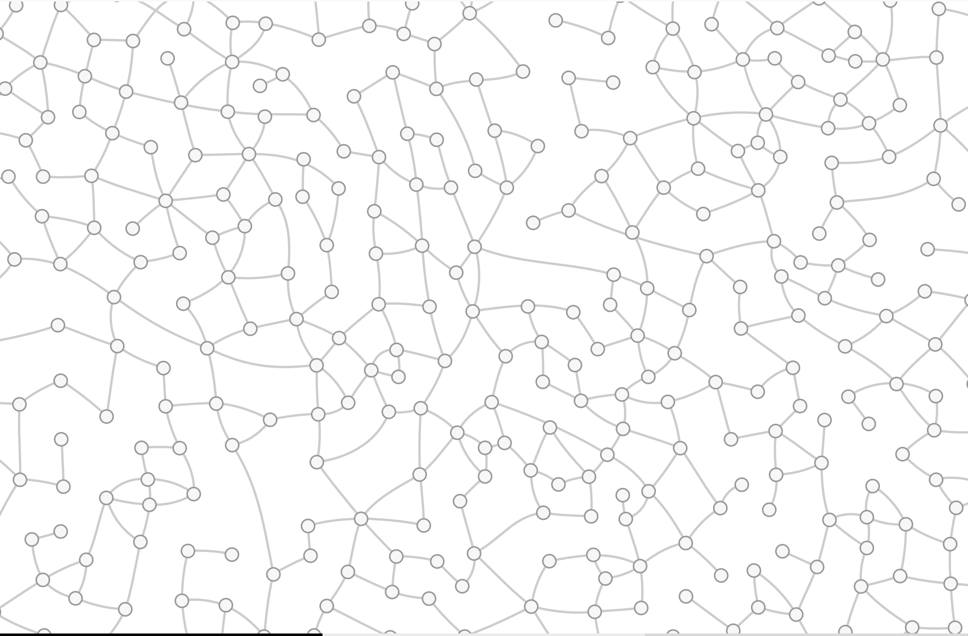
Lower Bounds *for* Local Approximation

Mika Göös, Juho Hirvonen & Jukka Suomela
University of Helsinki

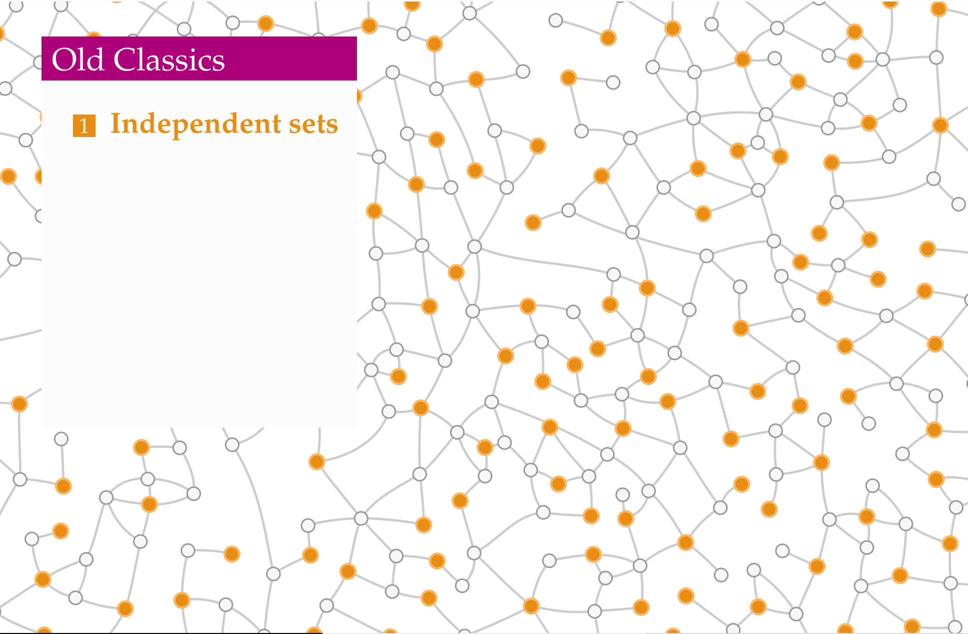
Lower Bounds *for* Local Approximation

Main Result: **Local algorithms** do not need ***unique IDs*** when solving graph optimization problems

Input = Graph $\mathcal{G}$ = Communication Network



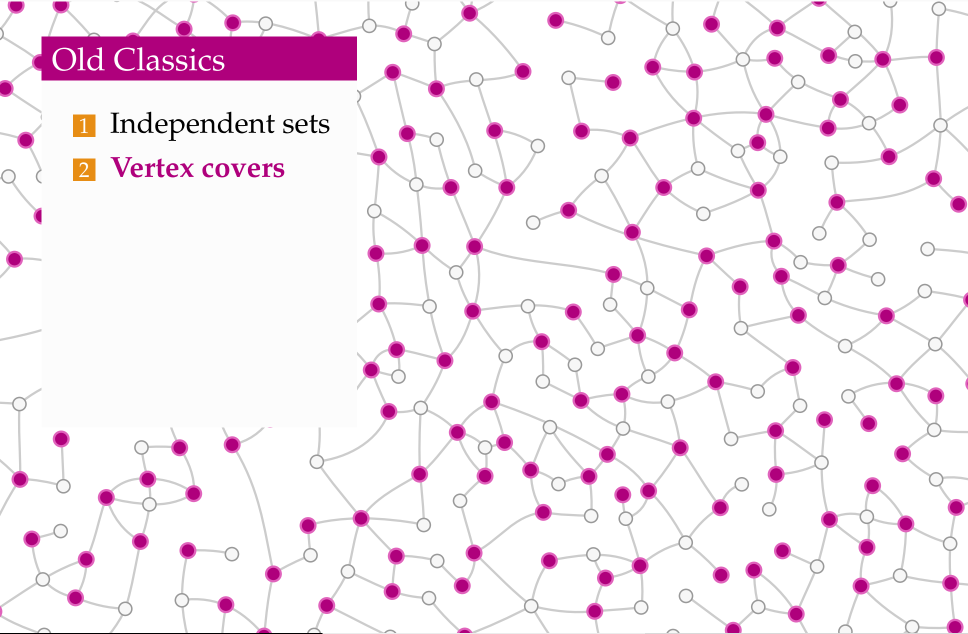
“Simple” Graph Problems



Old Classics

1. **Independent sets**

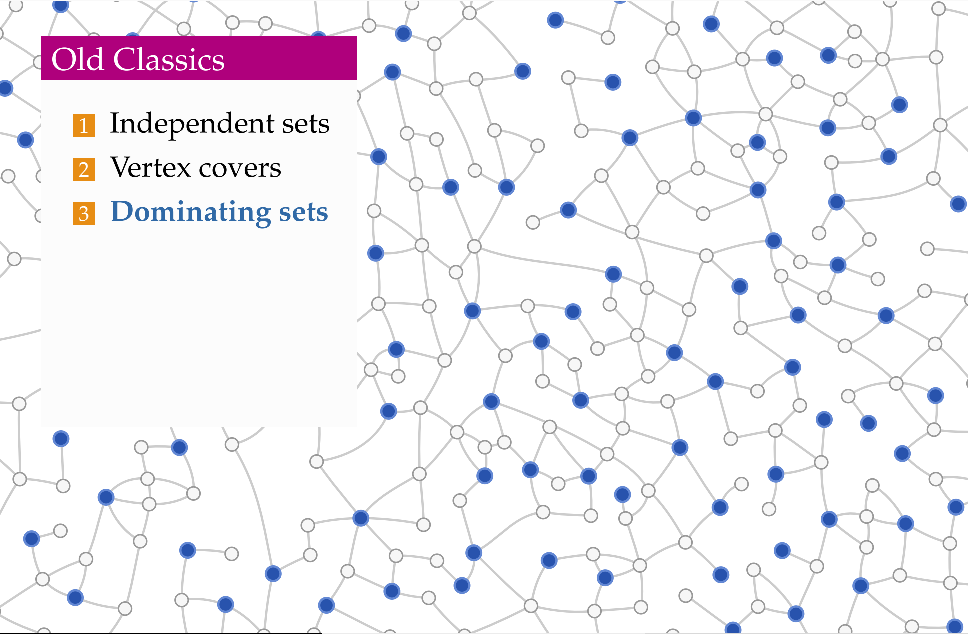
“Simple” Graph Problems



Old Classics

1. Independent sets
2. **Vertex covers**

"Simple" Graph Problems



Old Classics

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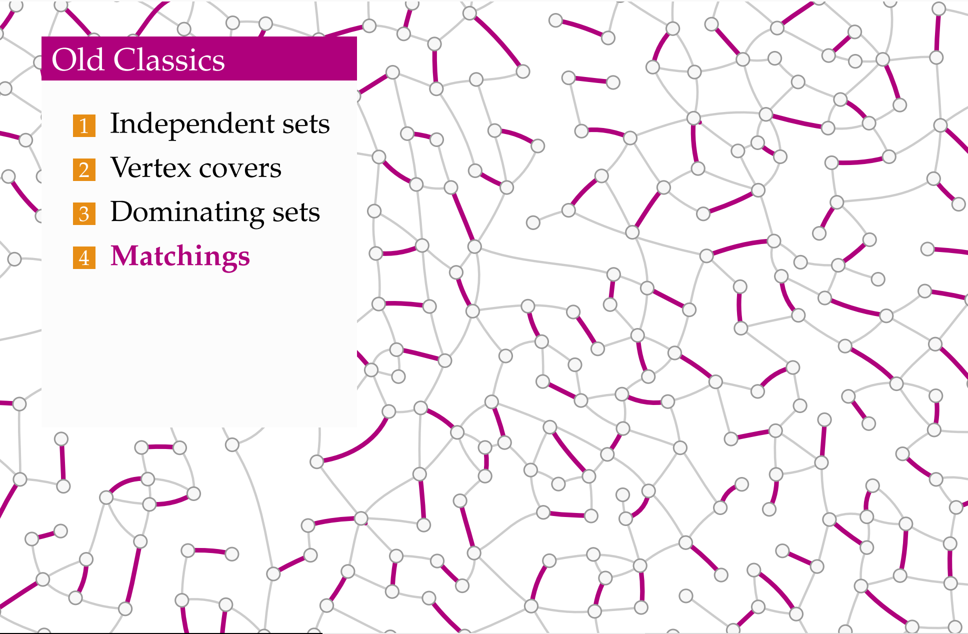
3. **Dominating sets**

“Simple” Graph Problems



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"Simple" Graph Problems

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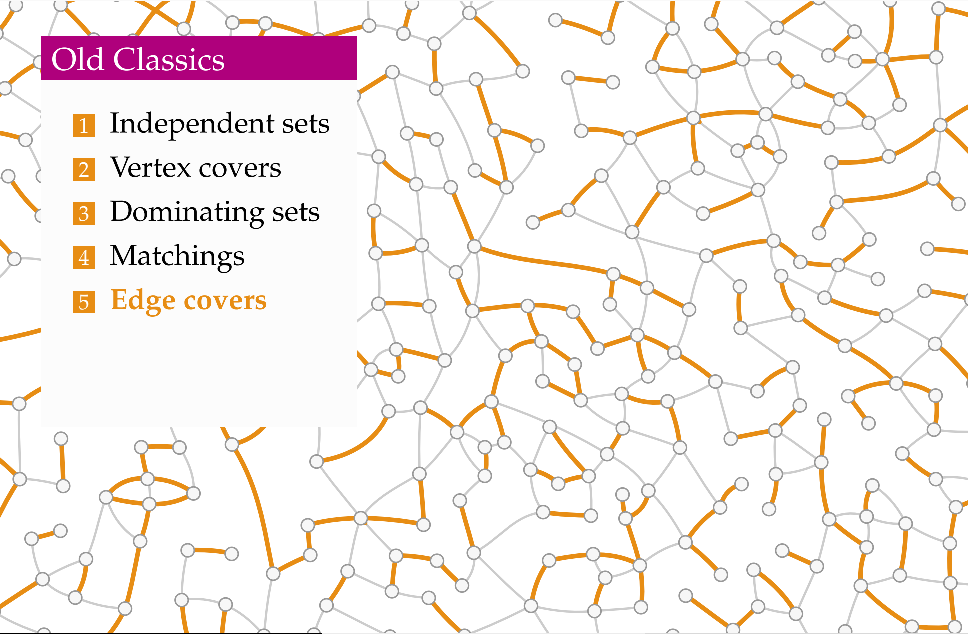
5. **Edge covers**



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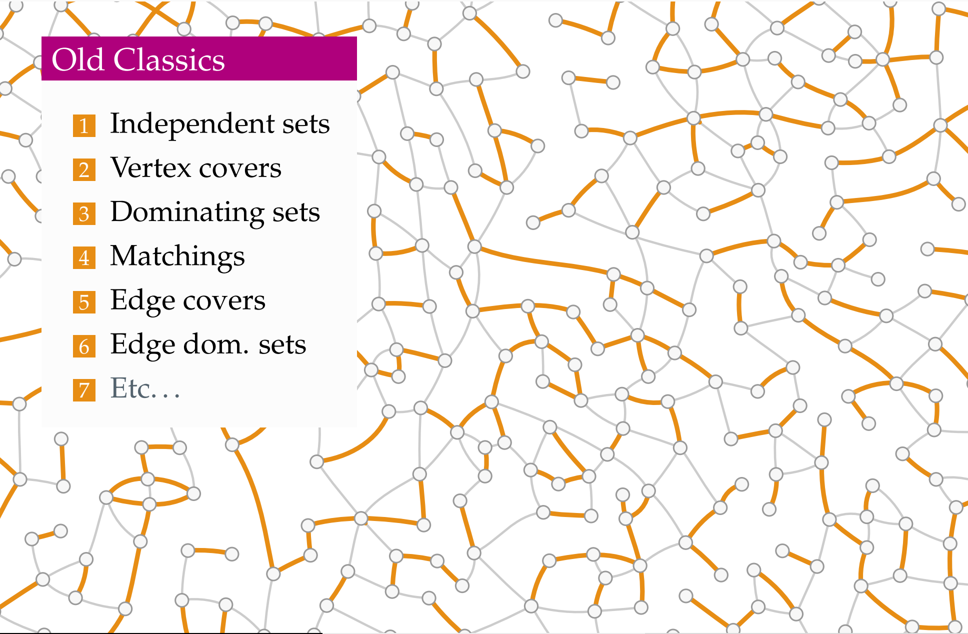

Old Classics

1. Independent sets
2. Vertex covers
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4. Matchings
5. Edge covers
6. Edge dom. sets
7. Etc. . .

Local Algorithms

1. Distributed algorithm **A**
2. Deterministic, synchronous

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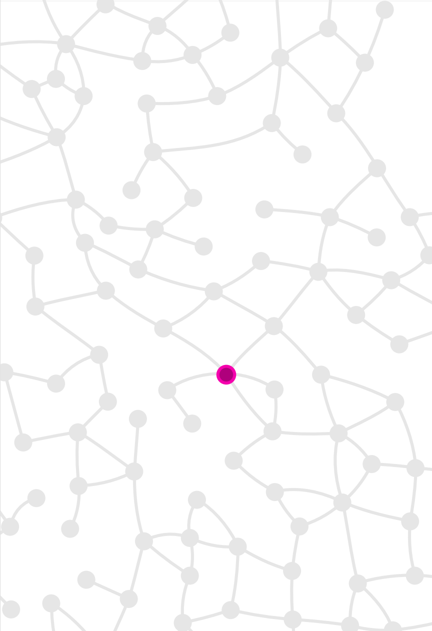

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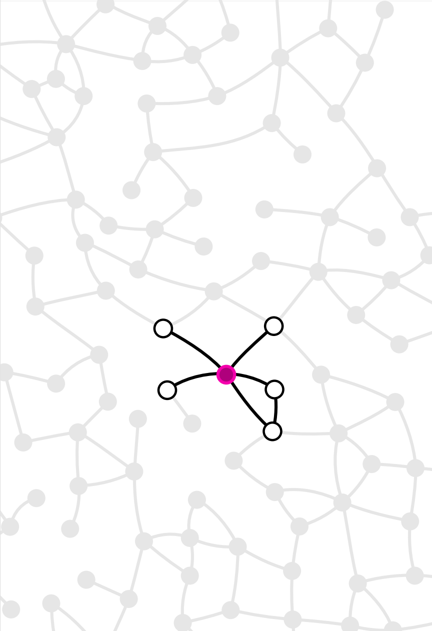

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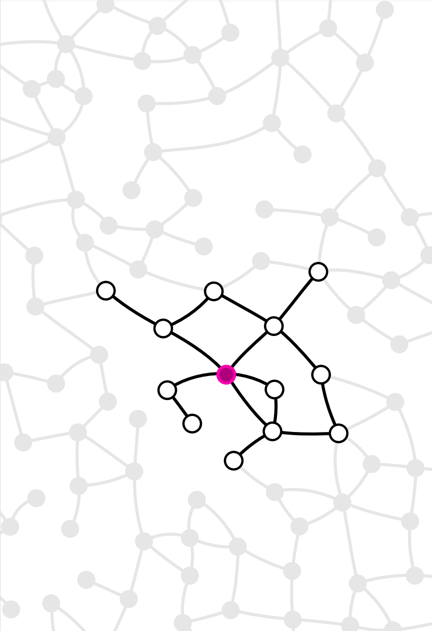

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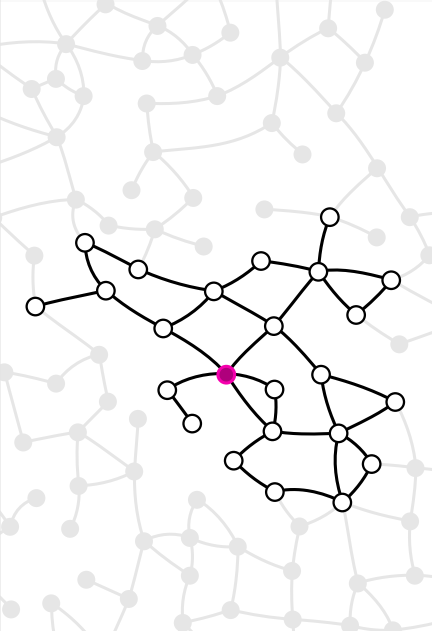

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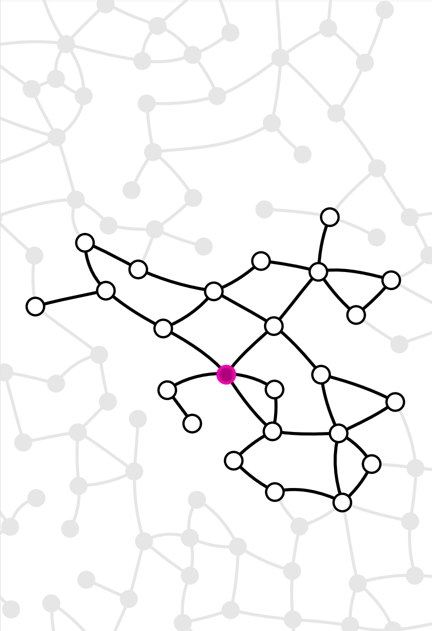

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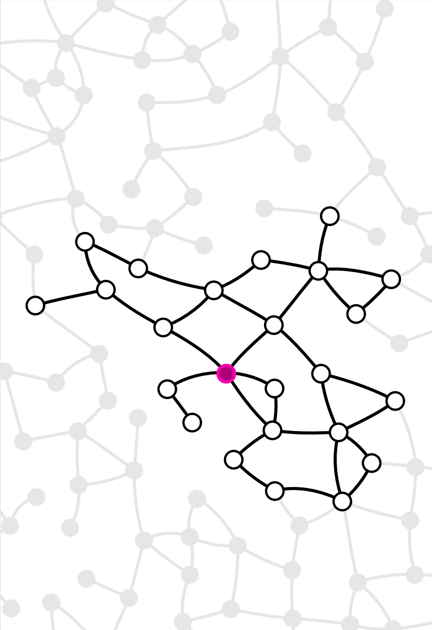
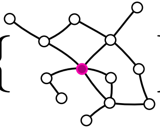
On ***bounded degree graphs*** ($\Delta = O(1)$) running time is a constant:

$$r \in \mathbb{N} \qquad (\text{e.g., } r = \mathbf{3})$$



Local Algorithms

Definition:

$\mathbf{A} : \{$  $\} \to \{0,1\}$

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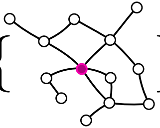
$\mathbf{A} : \{$  $\} \to \{0,1\}^{\Delta}$

Two Network Models

Unique Identifiers

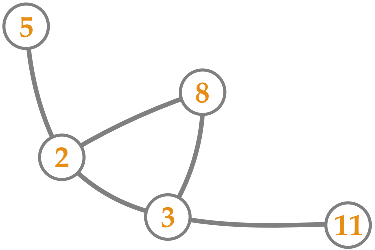
Anonymous Networks
with Port Numbering

Two Network Models

Unique Identifiers

- Each node has a unique $O(\log n)$-bit **label**:

$$V(\mathcal{G}) \subseteq \{1, 2, \ldots, \mathrm{poly}(n)\}$$



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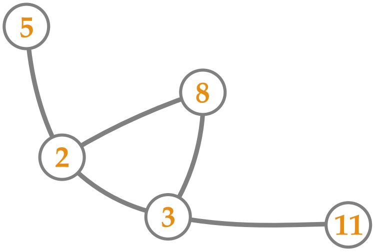
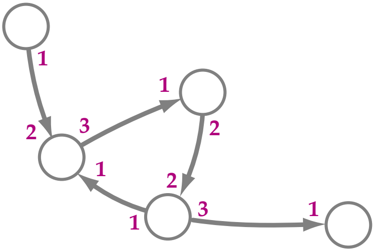


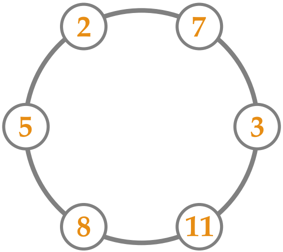
Anonymous Networks
with Port Numbering

- Node v can refer to its neighbours via **ports** $1, 2, \dots, \deg(v)$
- Edges are **oriented**

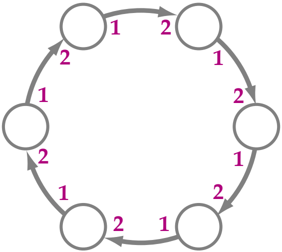


Example: Independent Sets on a Cycle

ID-model



PO-model



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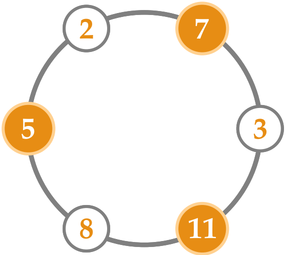

- [Cole–Vishkin 86, Linial 92]: **Maximal independent set** can be computed in $\Theta(\log^* n)$ rounds

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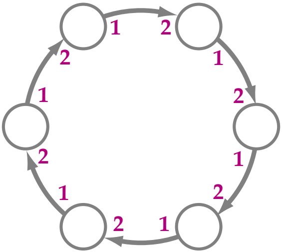

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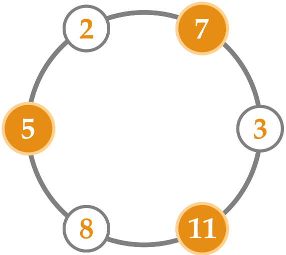

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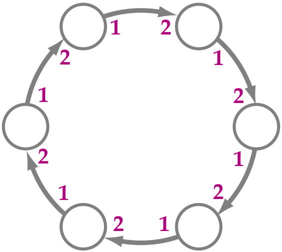

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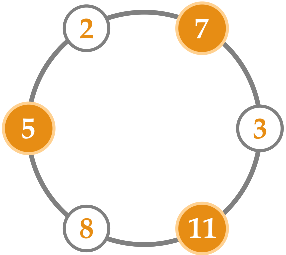

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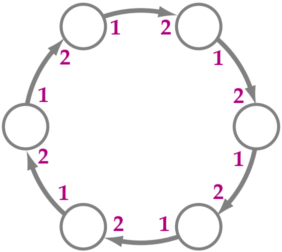

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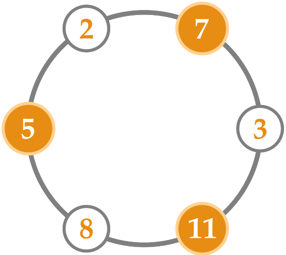

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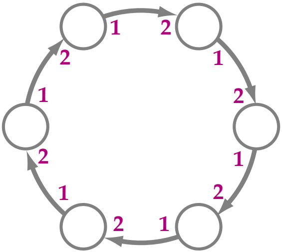

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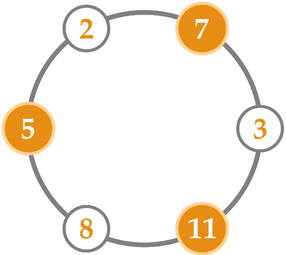
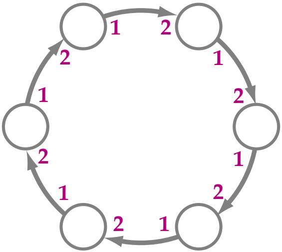
ID-model $\gg$ **in general** **PO-model**





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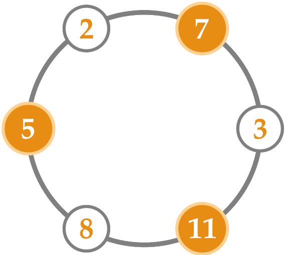

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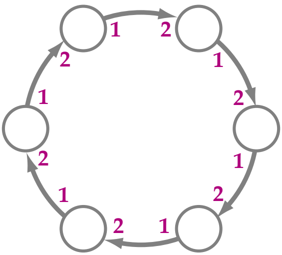

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$\approx$

for Local Approximation

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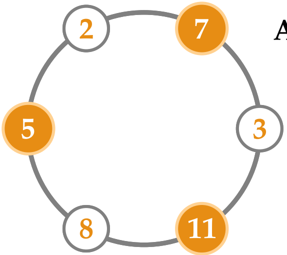

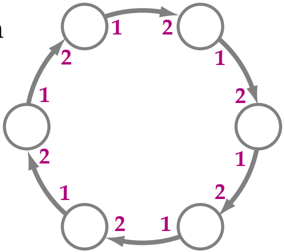

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Known Approximation Ratios

	ID	PO	
Max **Independent Set**	∞	∞	[LW **DISC'08**]
Max **Matching**	∞	∞	[CHW **DISC'08**]
Min **Vertex Cover**	2	2	[ÅS **SPAA'10**]
Min **Edge Cover**	2	2	
Min **Dominating Set**	$\Delta + 1$	$\Delta + 1$	
Min **Edge Dominating Set**	???	$4 - \frac{2}{\Delta}$	[Suomela **PODC'10**]

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Our Result (informally)

Main Thm: When **Local Algorithms** compute **constant factor** approximations,

$$\mathbf{ID} = \mathbf{PO}$$

for a general class of graph problems

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next: *proof idea...*

What is known?

ID **PO**

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Local ID-algorithms can only **compare** identifiers

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ID = OI PO

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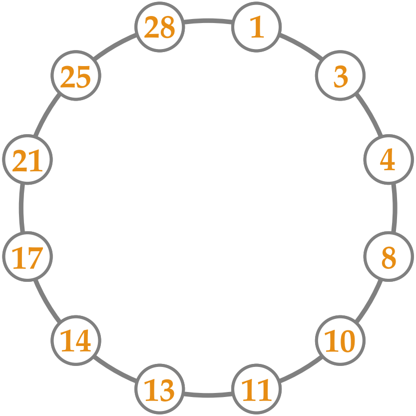
Order invariant (OI) algorithm A:

Input: Ordered graph $(\mathcal{G}, \leq)$
Output: $\mathbf{A}(\mathcal{G}, \leq, v)$ depends only on **order type** of the radius-r neighbourhood of v

What is known?

ID = OI PO



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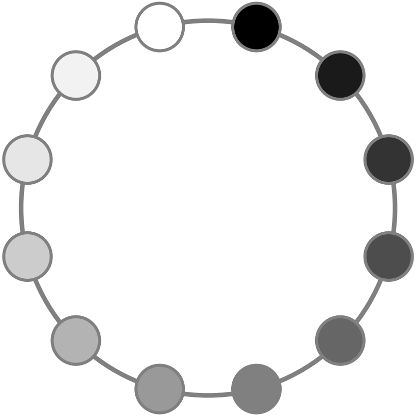

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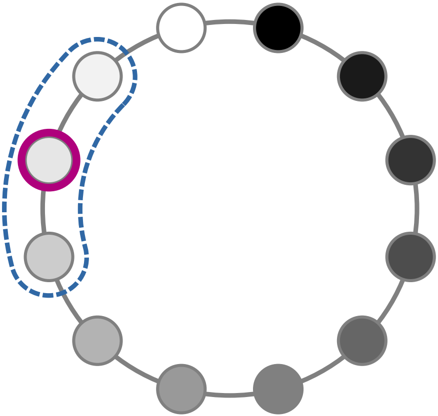

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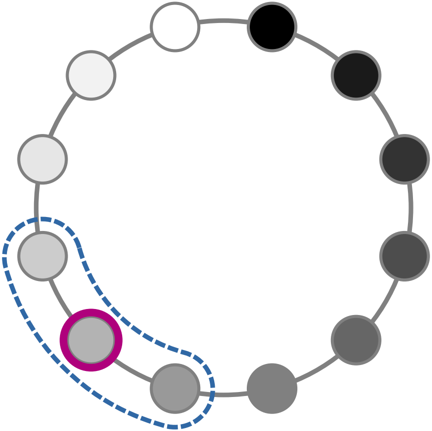

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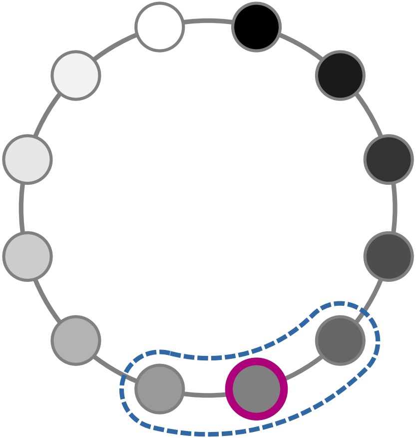

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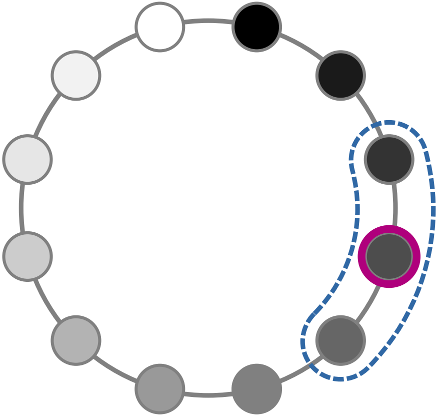

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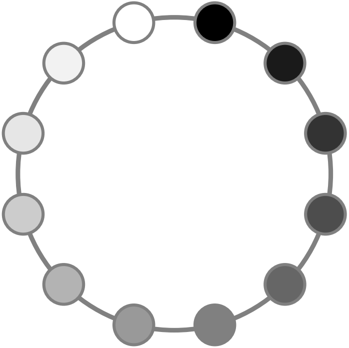

Ordered cycle:

$(1-\epsilon)$-**fraction** of neighbourhoods are isomorphic

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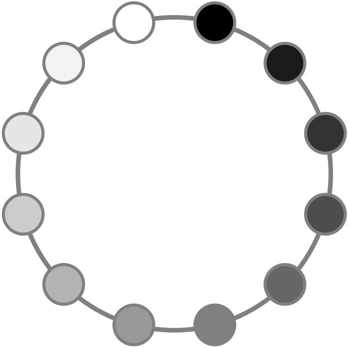

Ordered cycle:

$(1-\epsilon)$**-fraction** of neighbourhoods are isomorphic

$\Updownarrow$

$(1-\epsilon)$**-homogeneous**

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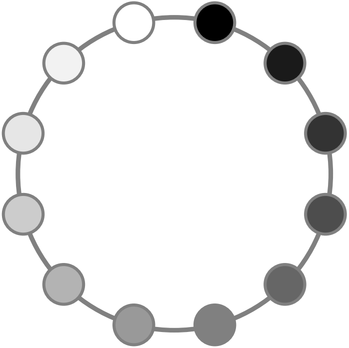
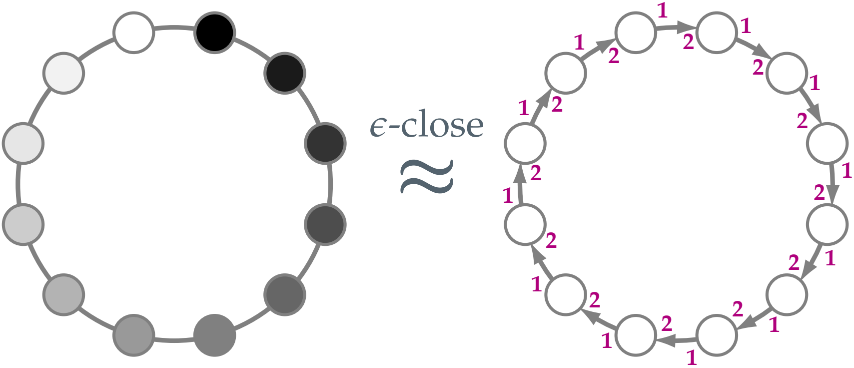


What is known?

$$\mathrm{ID} = \mathrm{OI} \stackrel{?}{=} \mathrm{PO}$$



Key Construction

High-degree homogeneous graphs:

(analogue to *homogeneously ordered cycles*)

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Algebraic construction:

We use **Cayley graphs** of **soluble groups**

Key Construction

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Proof of Main Thm:

Form graph products $(\mathcal{H}_\epsilon, \leq_\epsilon) \times \mathcal{G}$

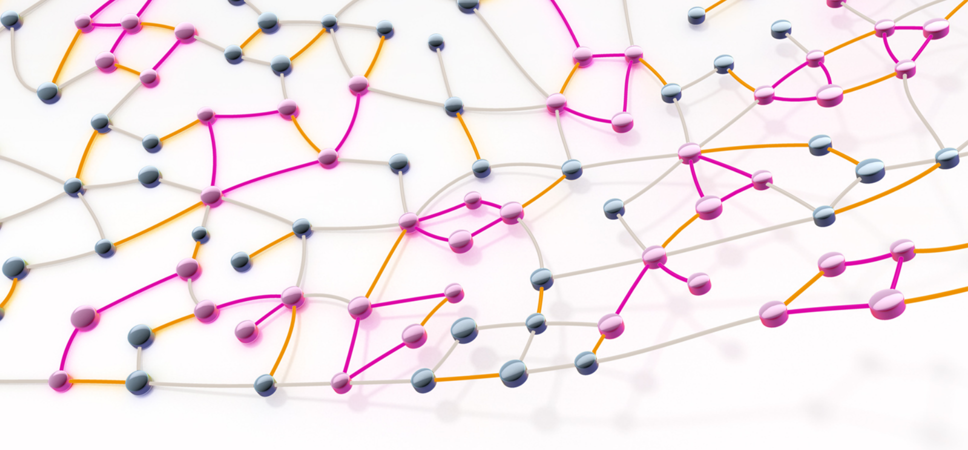
Our result:

For Local Approximation,

$$\mathbf{ID} = \mathbf{OI} = \mathbf{PO}$$

Open problems:

- Planar graphs?
- More applications for **homogeneous graphs**?
- Randomness!



Cheers!