Jukka Suomela
Aalto University

Limits of Distributed Computing

Computer network

Computer network

- exchange information with your neighbors

Computer network

- exchange information with your neighbors
- update your own state

Computer network

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- update your own state
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Computer network

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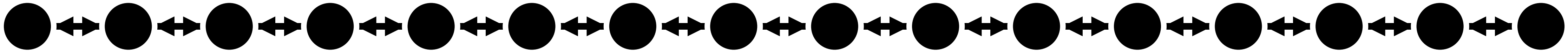
- update your own state
- produce your final output (e.g. color)



- exchange information with your neighbors
- update your own state
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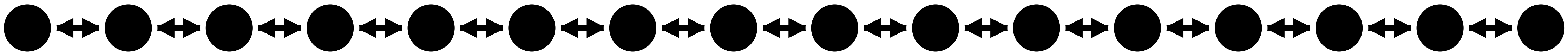
***T* = 3 rounds**

What can we say about final outputs after *T* rounds?



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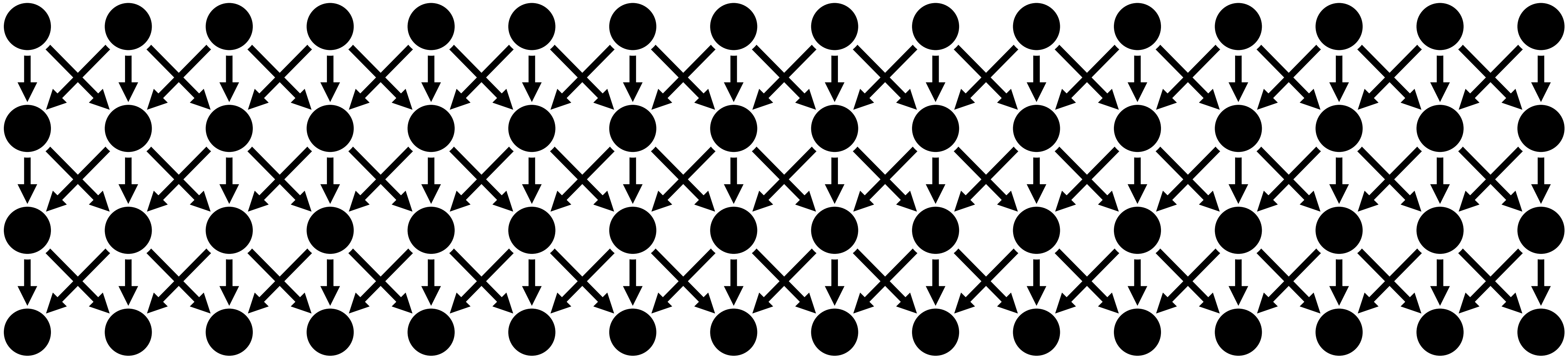
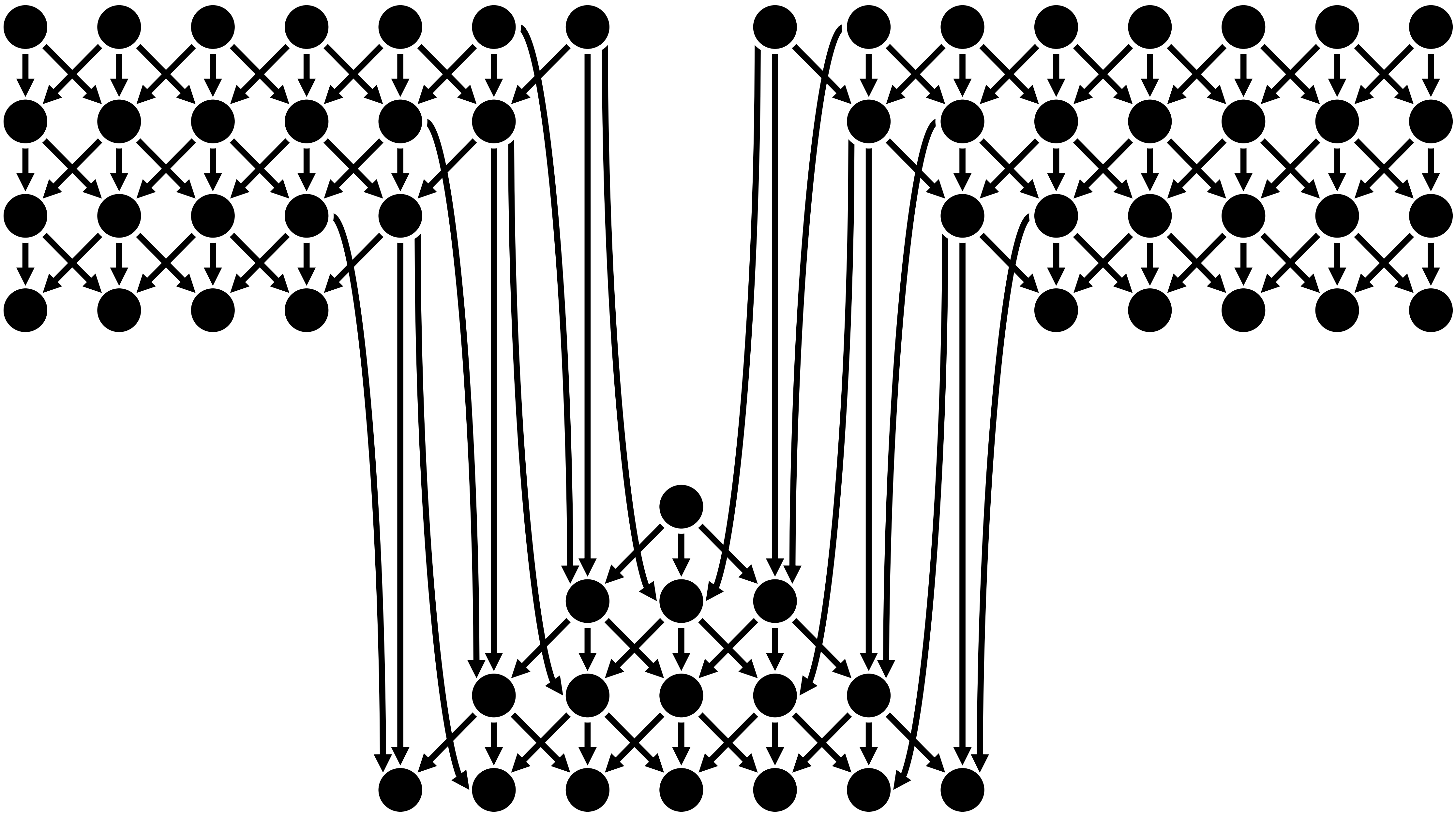
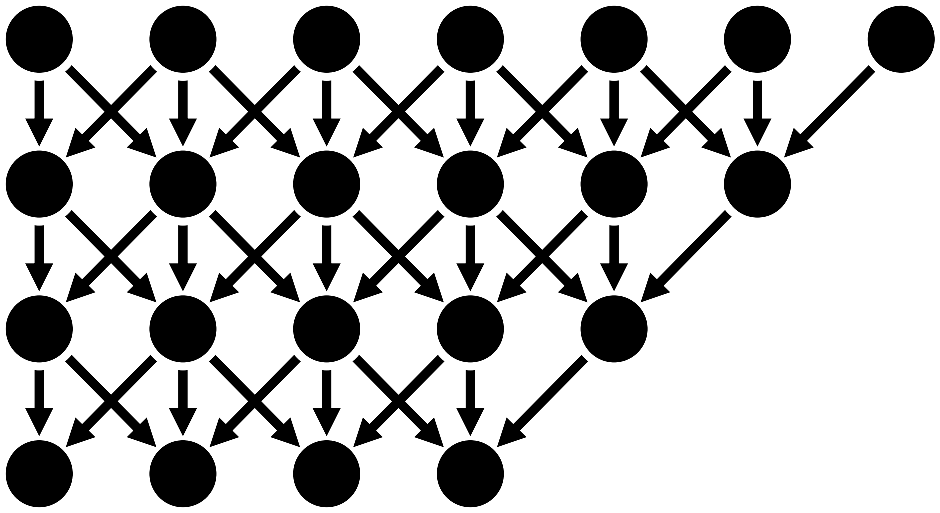
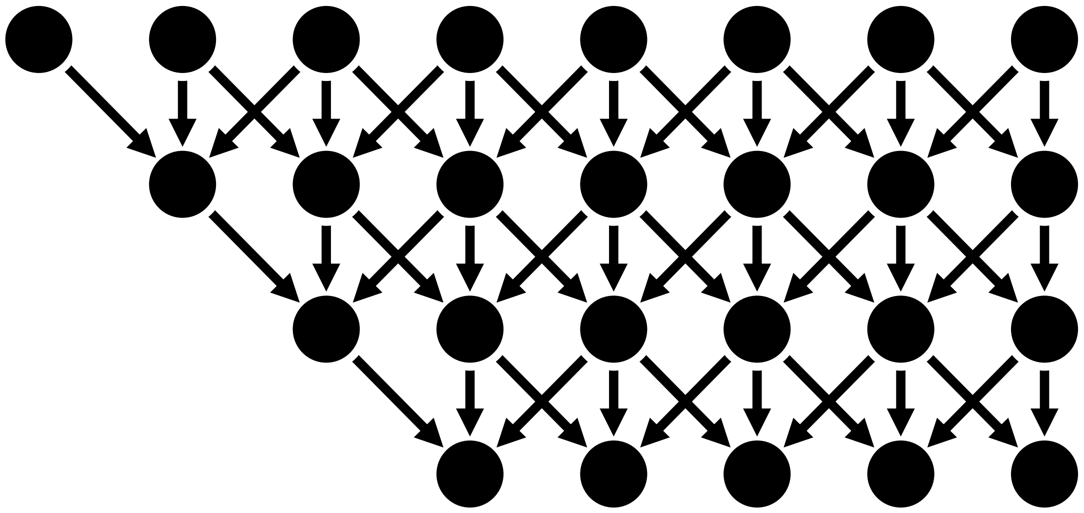
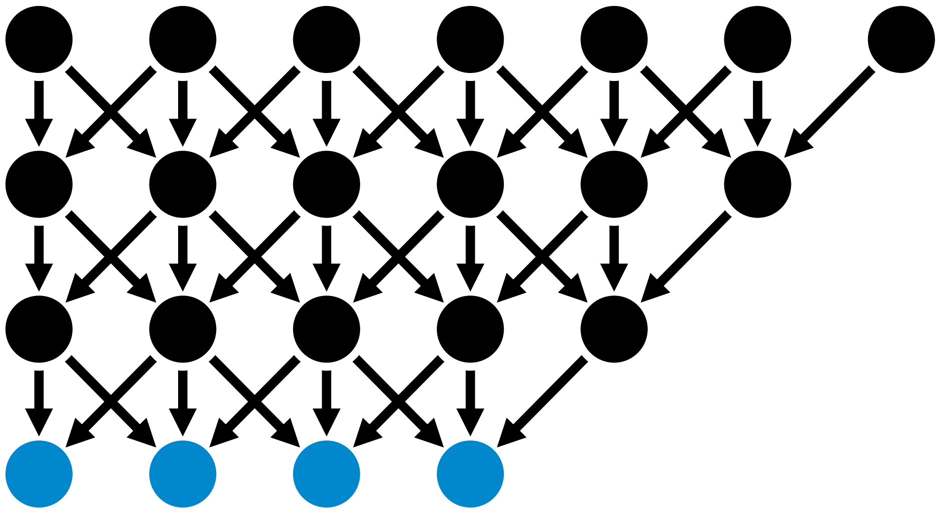


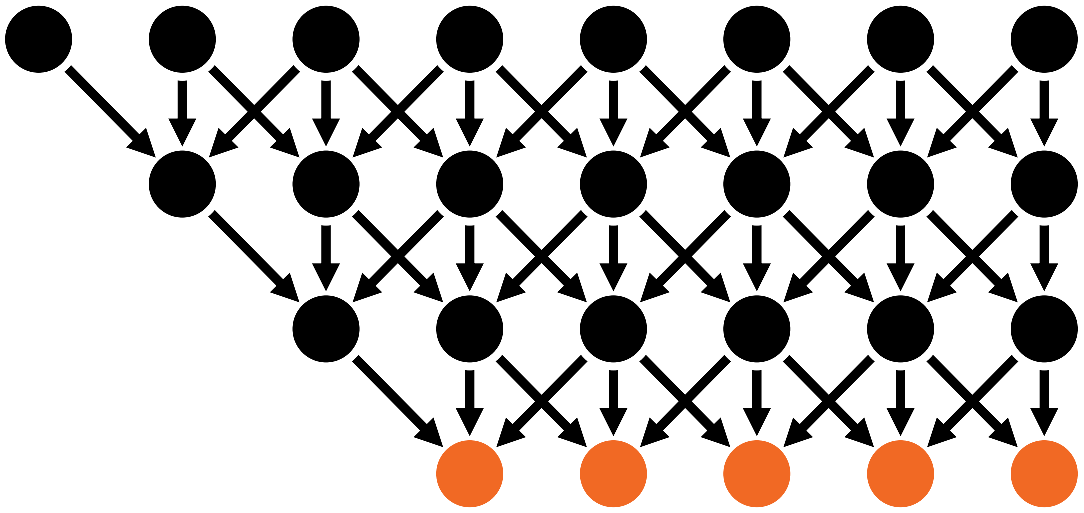








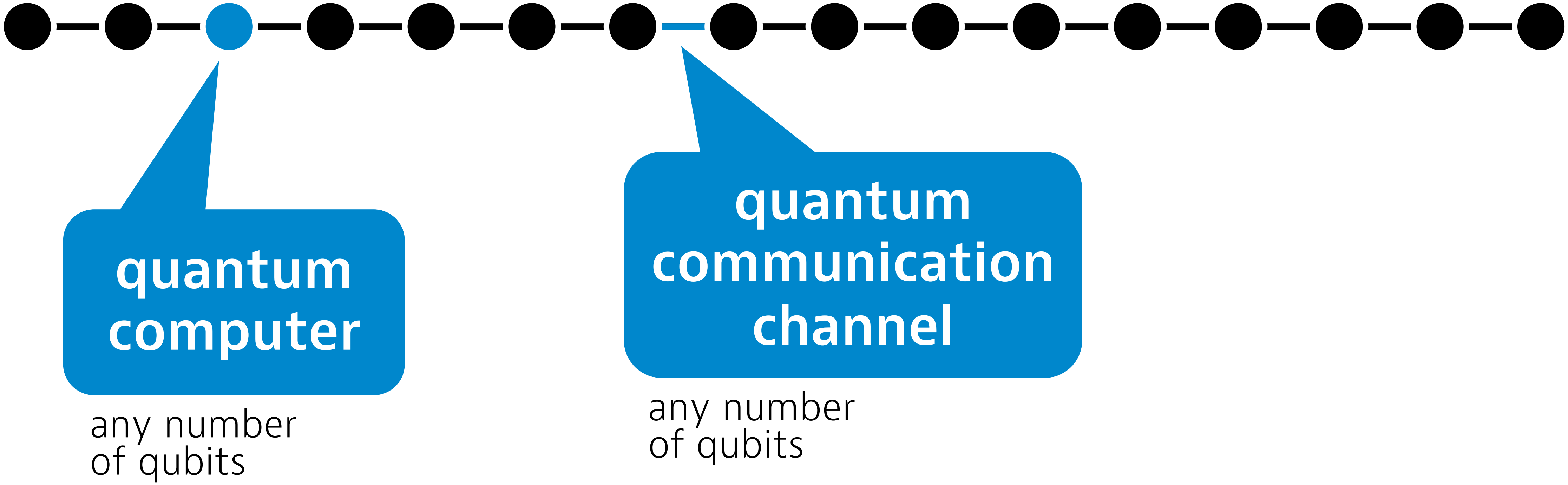
Whatever
these nodes
output …



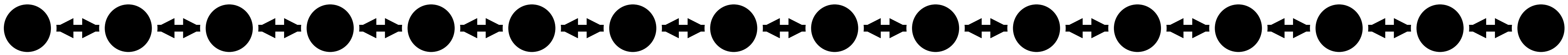
… is independent
of whatever
these nodes
output

What if it's quantum?

Network of quantum computers

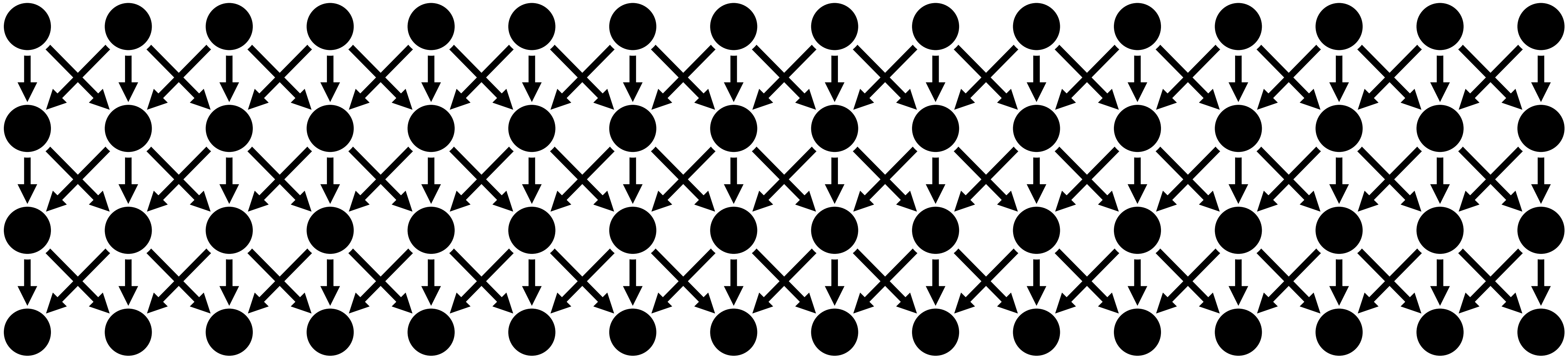


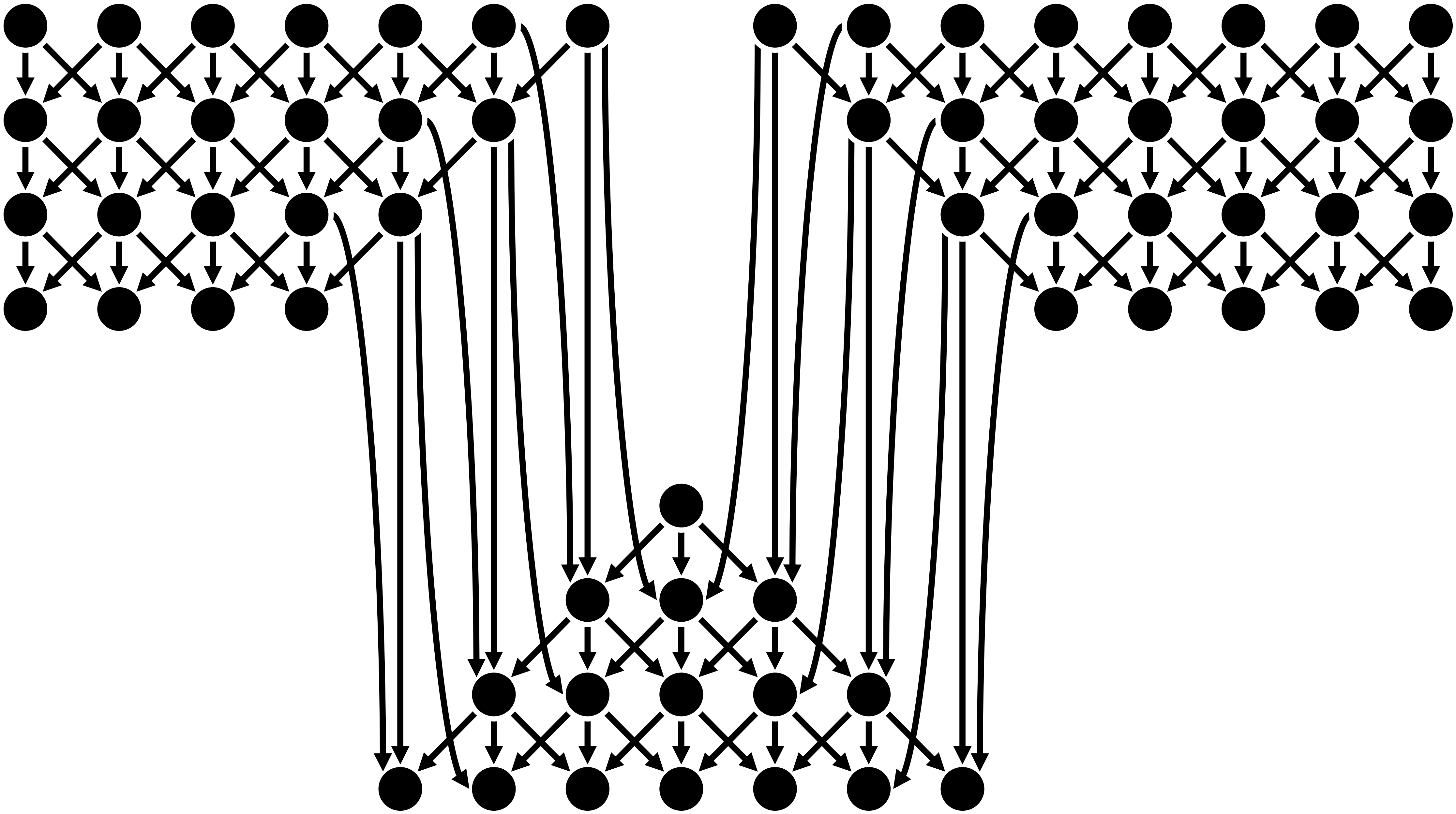
What can we say about final outputs after *T* rounds?

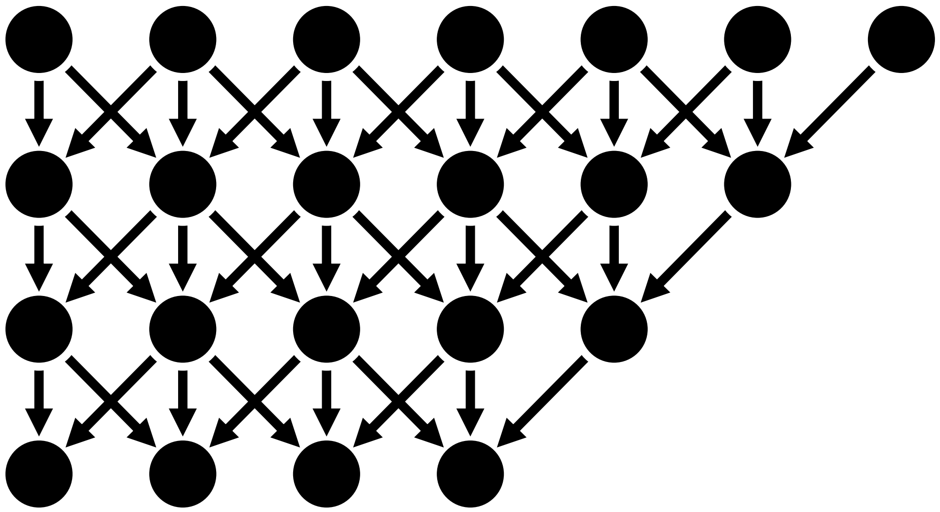


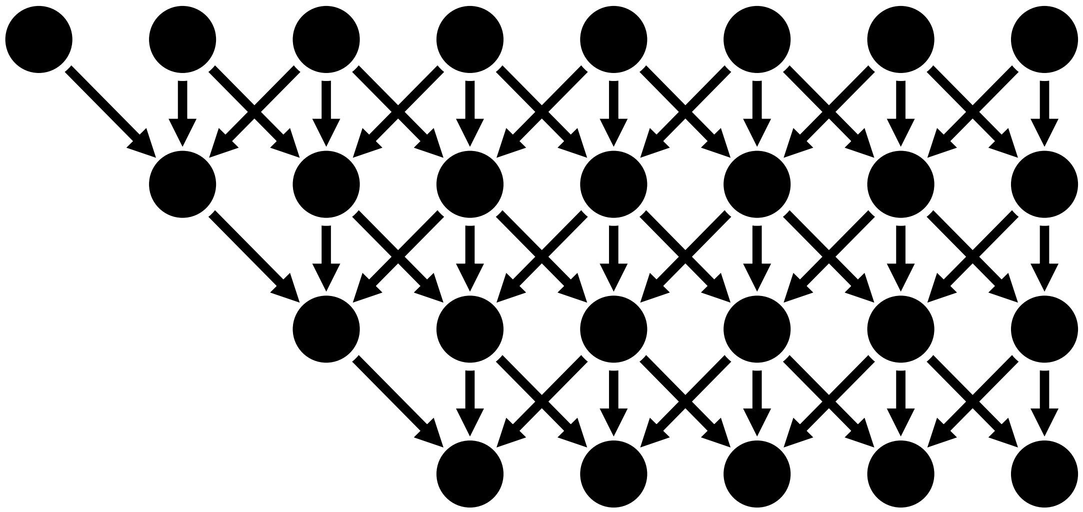
- exchange **qubits** with your neighbors
- update your own **quantum** state
- exchange **qubits** with your neighbors
- update your own **quantum** state
- exchange **qubits** with your neighbors
- update your own **quantum** state
- produce your final output (e.g. color)

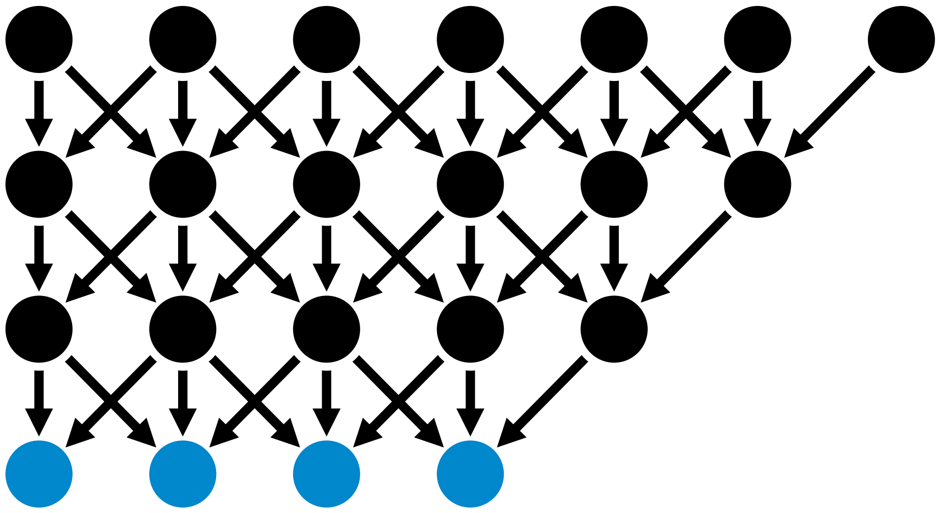
T = 3 rounds



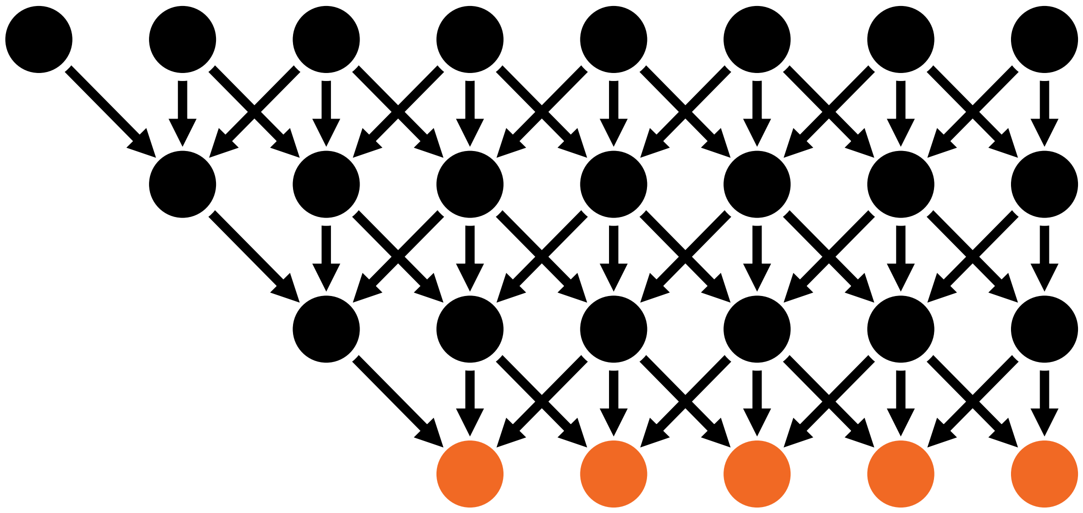






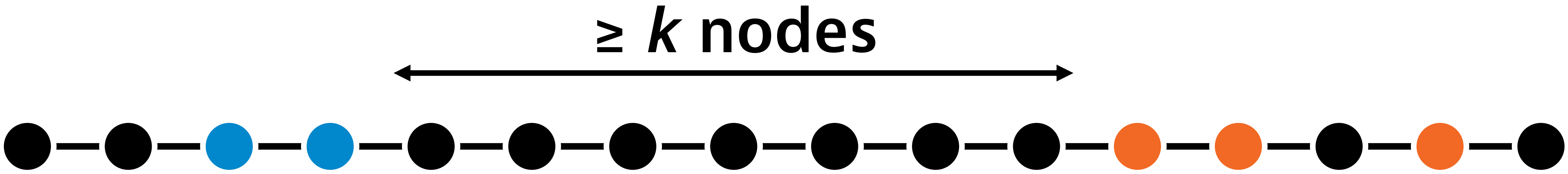


Whatever these nodes output ...



... is independent of whatever these nodes output

k-dependent distribution:



These random variables…

… are independent of these random variables

Finitely dependent distribution:

k-dependent for some constant *k*

Constant-round distributed algorithm
(classical or quantum!)

Finitely dependent distribution:
k-dependent for some constant *k*

Constant-round distributed algorithm

(classical or quantum!)

Show that these don’t exist for problem X …

Finitely dependent distribution:

k-dependent for some constant *k*

Constant-round distributed algorithm
(classical or quantum!)

... and we have impossibility here!

↓

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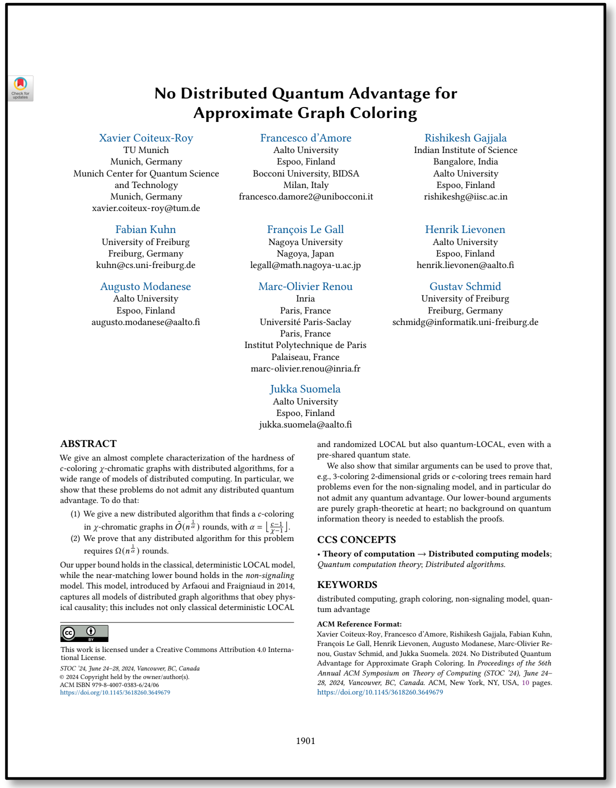
STOC 2024



No Distributed Quantum Advantage for Approximate Graph Coloring

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ABSTRACT

We give an almost complete characterization of the hardness of c-coloring χ-chromatic graphs with distributed algorithms, for a wide range of models of distributed computing. In particular, we show that these problems do not admit any distributed quantum advantage. To do that:

(1) We give a new distributed algorithm that finds a c-coloring in χ-chromatic graphs in $\tilde{O}(n^{\frac{1}{\alpha}})$ rounds, with $\alpha = \lfloor \frac{c-1}{\chi-1} \rfloor$.
(2) We prove that any distributed algorithm for this problem requires $\Omega(n^{\frac{1}{\alpha}})$ rounds.

Our upper bound holds in the classical, deterministic LOCAL model, while the near-matching lower bound holds in the *non-signaling* model. This model, introduced by Arfaoui and Fraigniaud in 2014, captures all models of distributed graph algorithms that obey physical causality; this includes not only classical deterministic LOCAL and randomized LOCAL but also quantum-LOCAL, even with a pre-shared quantum state.

We also show that similar arguments can be used to prove that, e.g., 3-coloring 2-dimensional grids or c-coloring trees remain hard problems even for the non-signaling model, and in particular do not admit any quantum advantage. Our lower-bound arguments are purely graph-theoretic at heart; no background on quantum information theory is needed to establish the proofs.

CCS CONCEPTS

• **Theory of computation → Distributed computing models**; *Quantum computation theory*; *Distributed algorithms.*

KEYWORDS

distributed computing, graph coloring, non-signaling model, quantum advantage

ACM Reference Format:
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Constant-round distributed algorithm
(classical or quantum!)

Finitely dependent distribution:
k-dependent for some constant *k*

Example: 3-coloring cycles

No constant-round classical algorithm for 3-colorings in cycles

[Linial 1992, Naor 1991]

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Finitely dependent distribution of 3-colorings in cycles

[Holroyd & Liggett 2016, Holroyd, Hutchcroft, Levy 2018]

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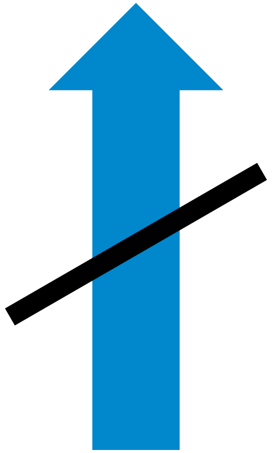

Finitely dependent distribution of 3-colorings in cycles

[Holroyd & Liggett 2016, Holroyd, Hutchcroft, Levy 2018]

No constant-round classical algorithm for 3-colorings in cycles

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Constant-round quantum algorithm for 3-coloring cycles???

???

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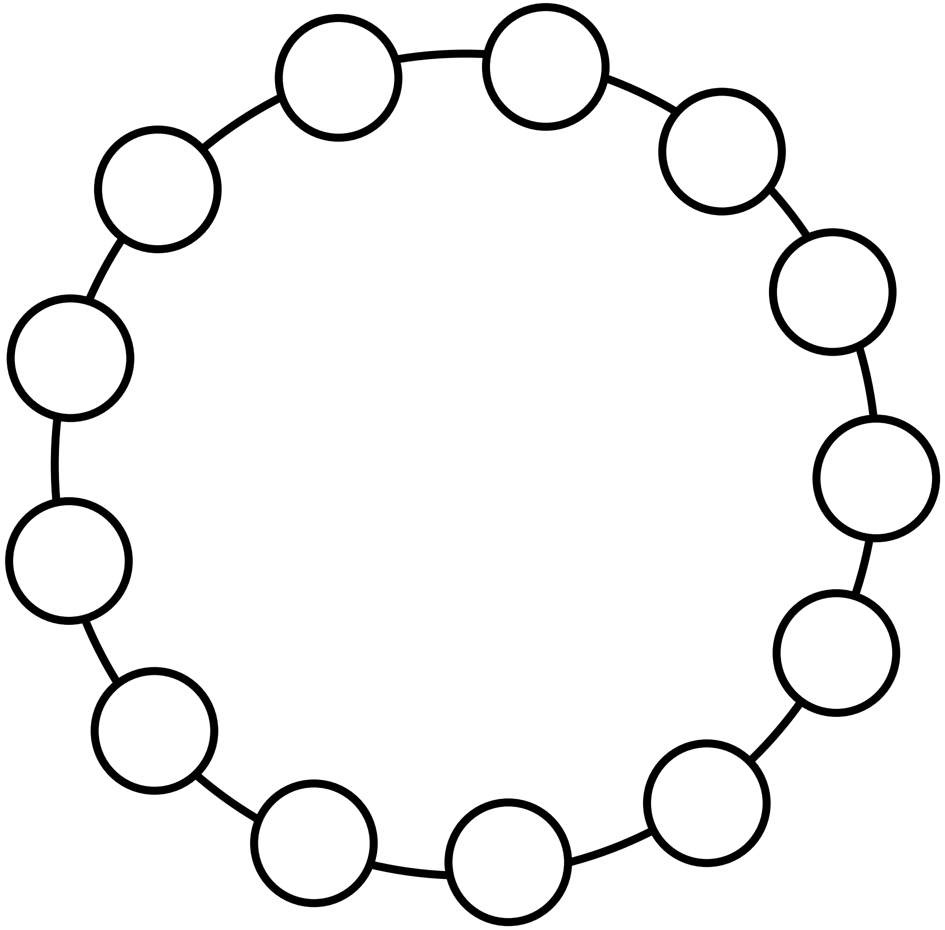
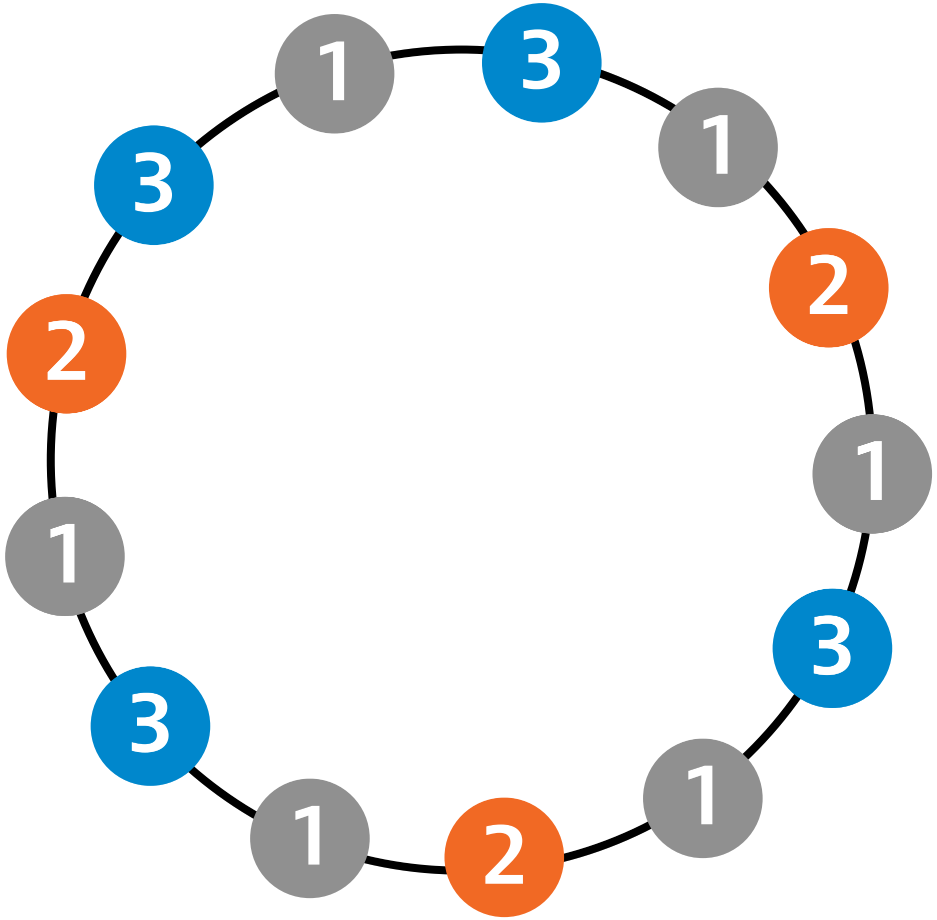
Given: n-cycle
(for any $n \geq 3$)



Finitely dependent distribution of 3-colorings in cycles

[Holroyd & Liggett 2016, Holroyd, Hutchcroft, Levy 2018]

Challenge:
design a random process that produces proper 3-colorings that are k-dependent?



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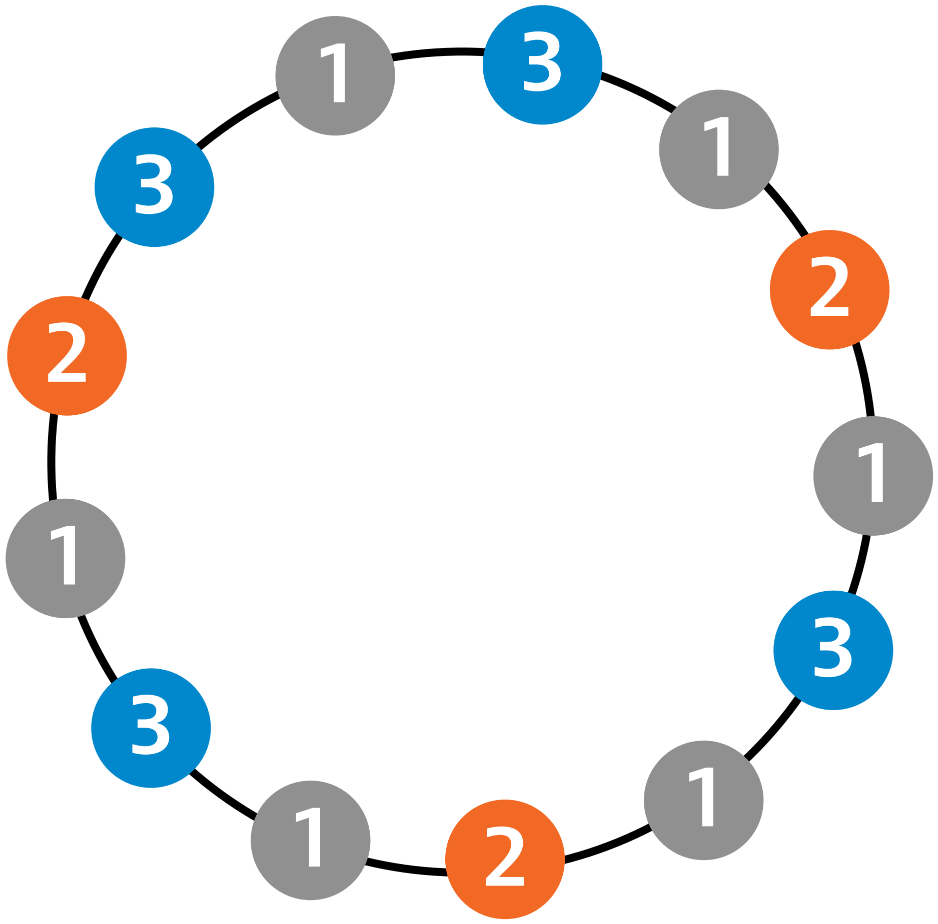

Uniformly random colorings do not work!
(exercise: why?)

Finitely dependent distribution of 3-colorings in cycles

[Holroyd & Liggett 2016, Holroyd, Hutchcroft, Levy 2018]

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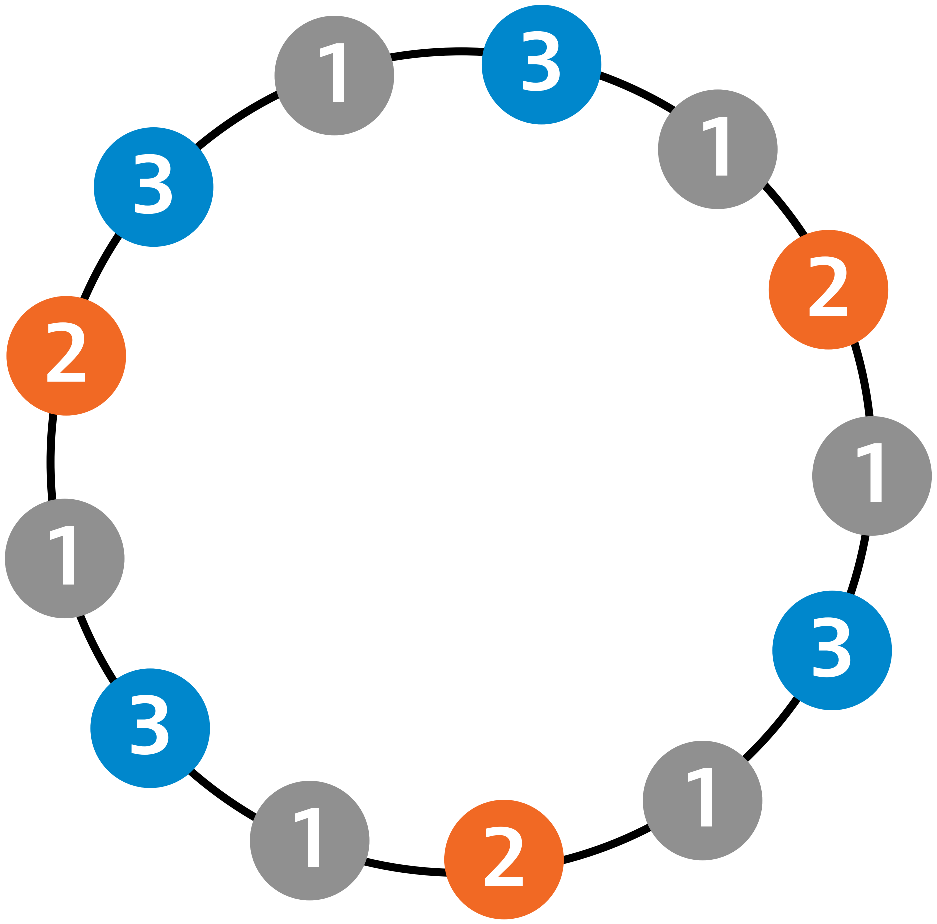
we will have k = 2



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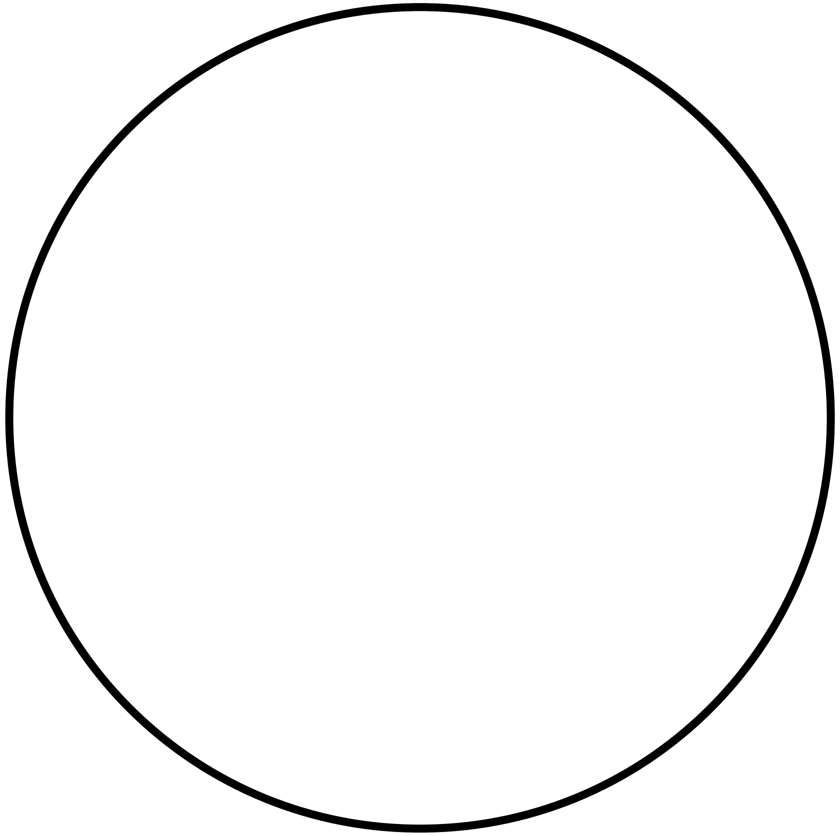
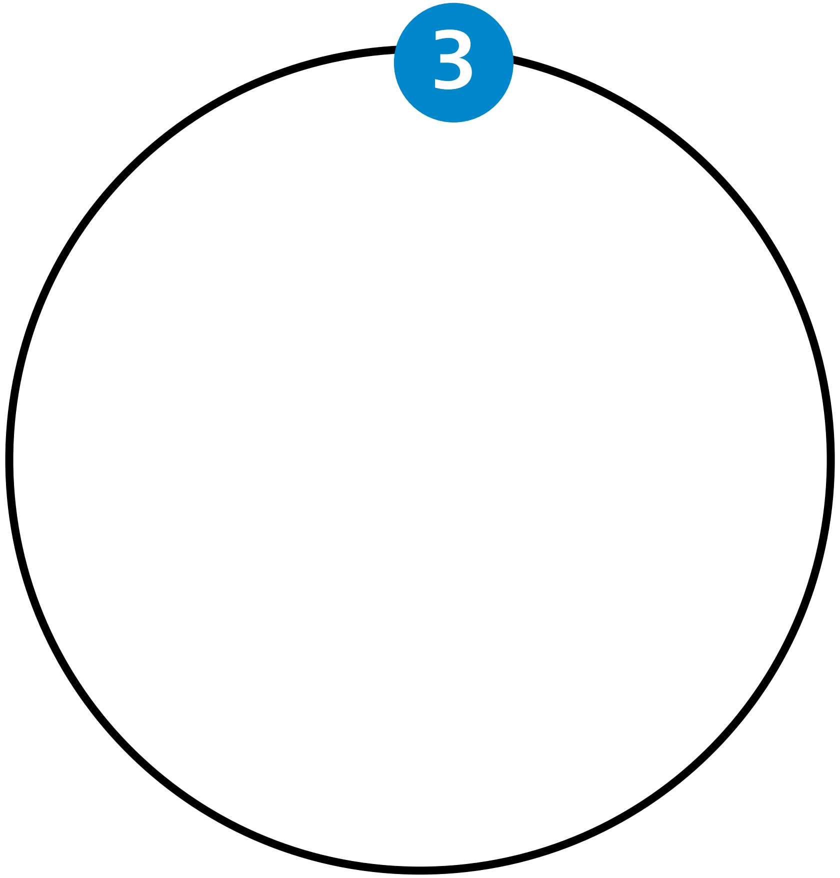
Finitely dependent distribution of 3-colorings in cycles

[Holroyd & Liggett 2016, Holroyd, Hutchcroft, Levy 2018]



Finitely dependent distribution of 3-colorings in cycles

[Holroyd & Liggett 2016, Holroyd, Hutchcroft, Levy 2018]



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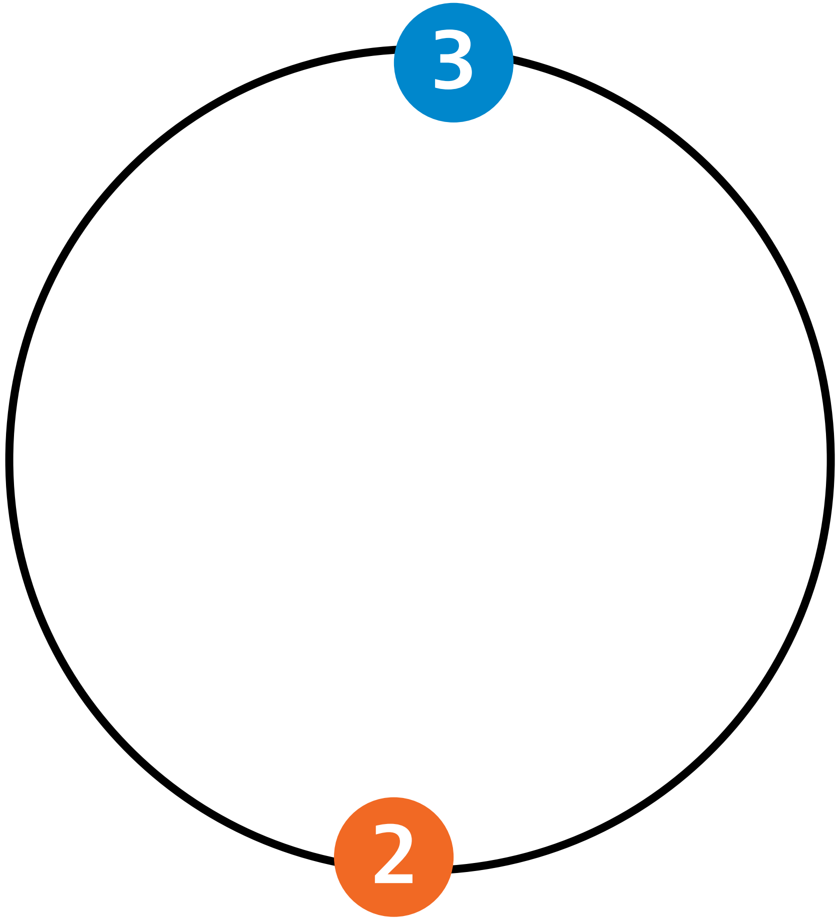
[Holroyd & Liggett 2016, Holroyd, Hutchcroft, Levy 2018]



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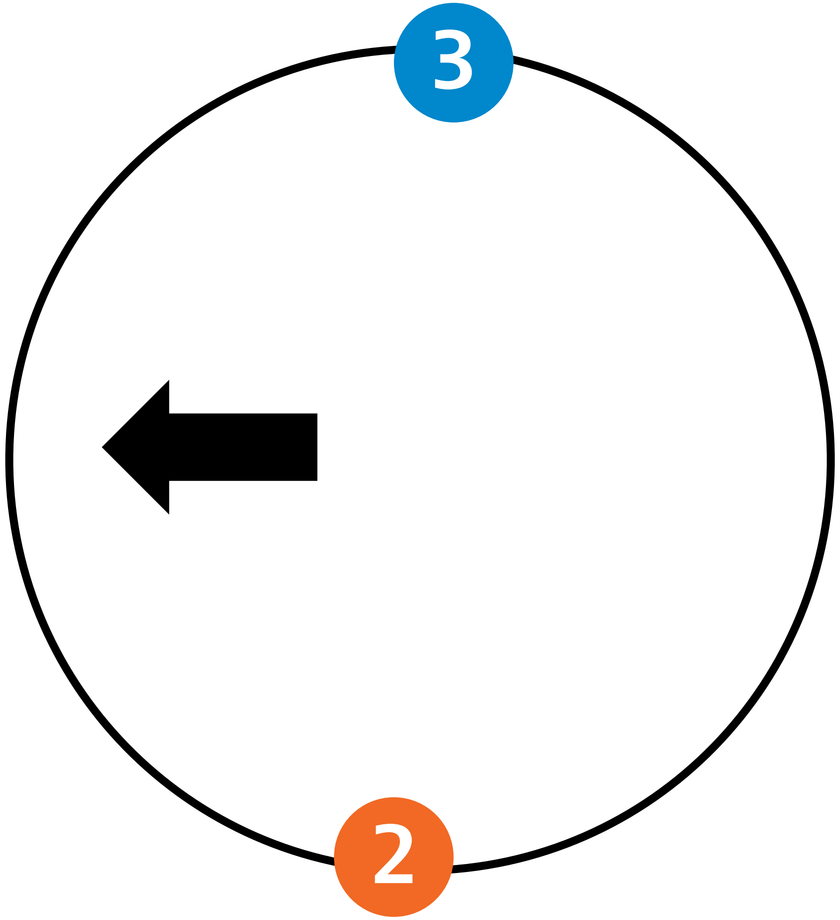

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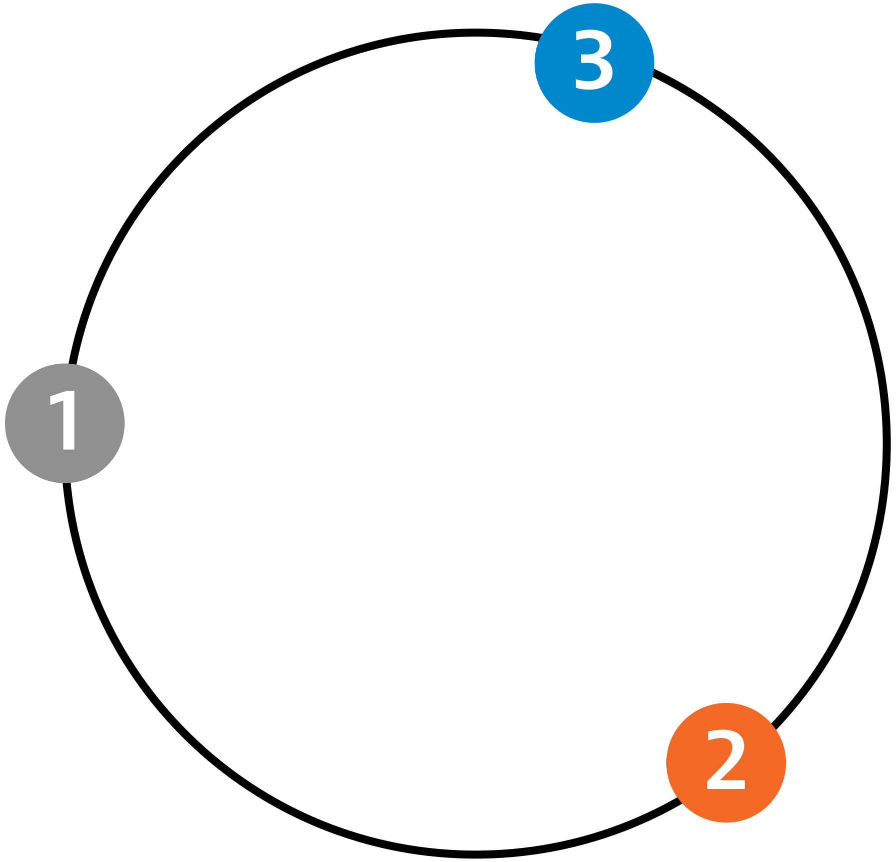

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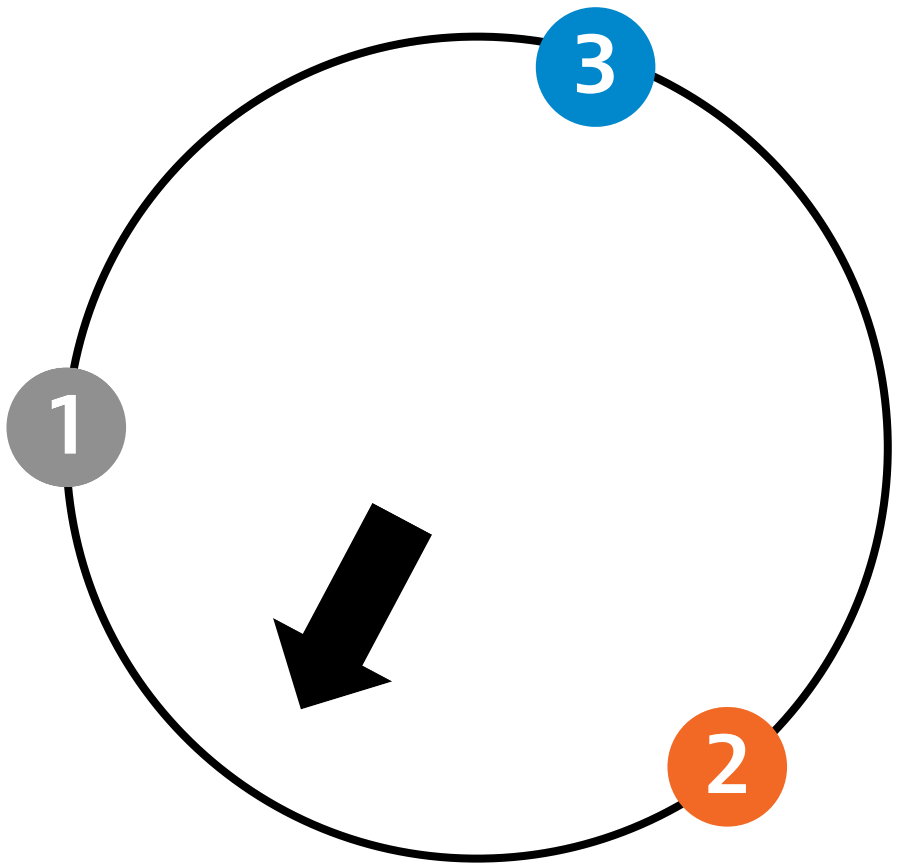

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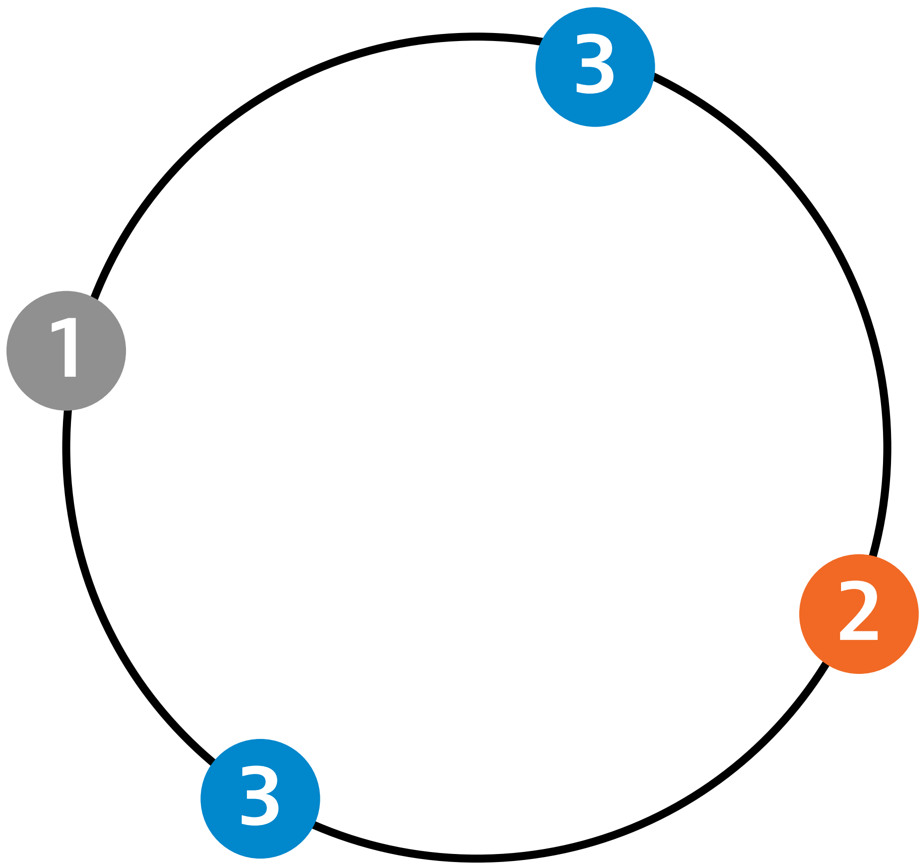

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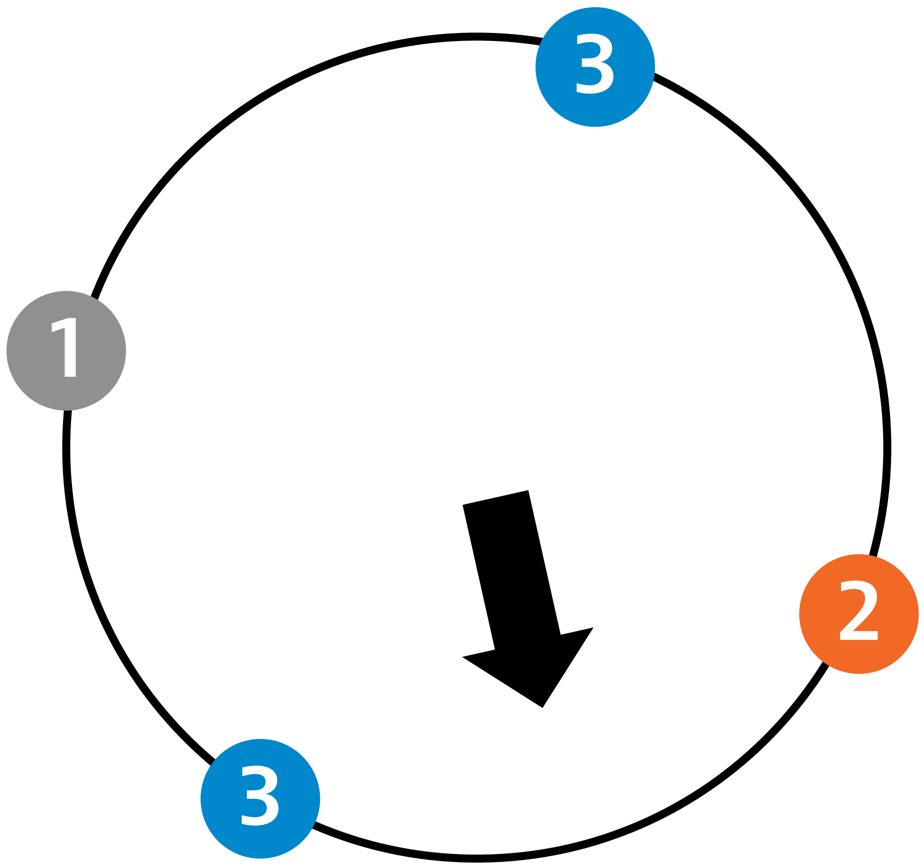

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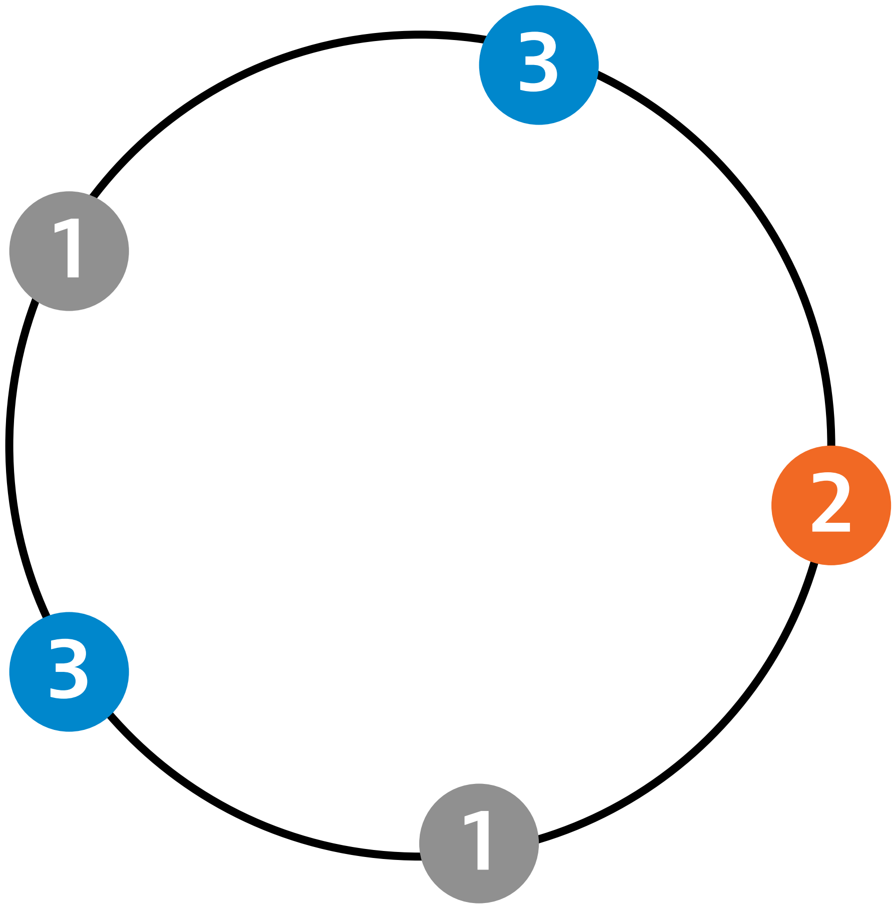

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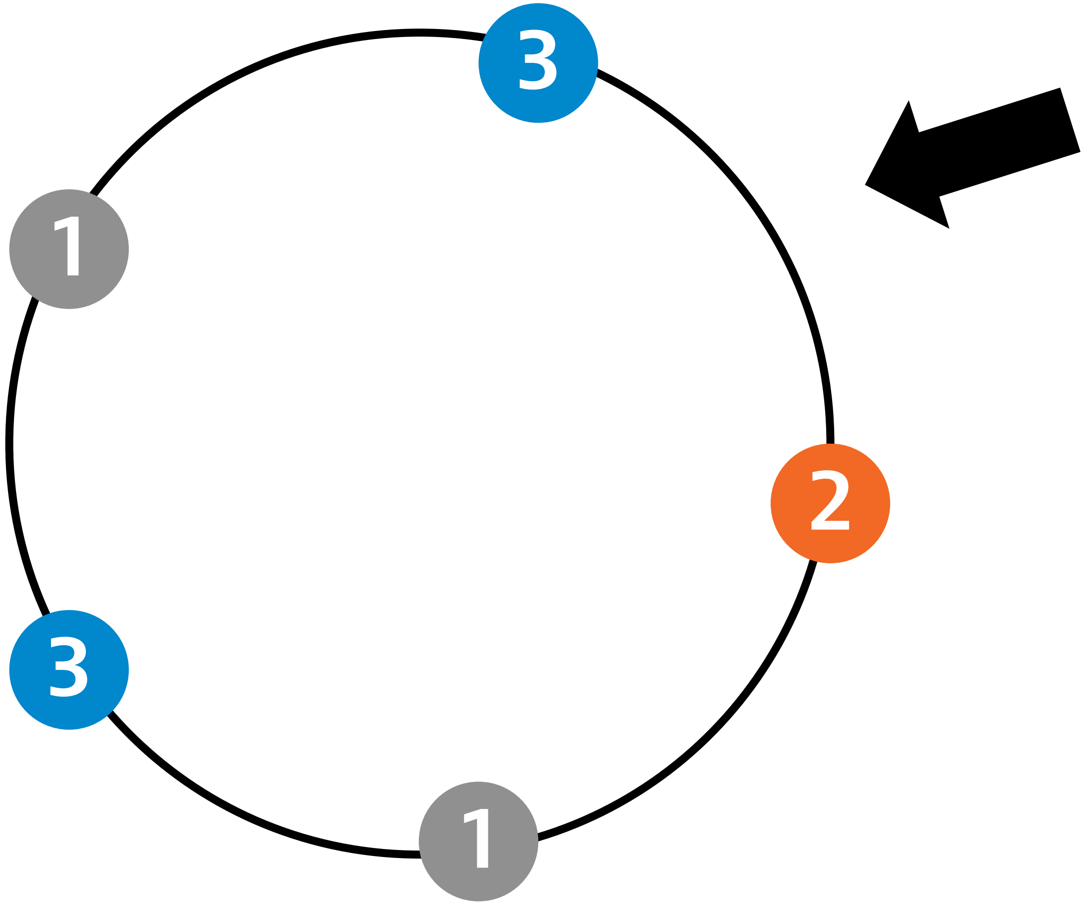

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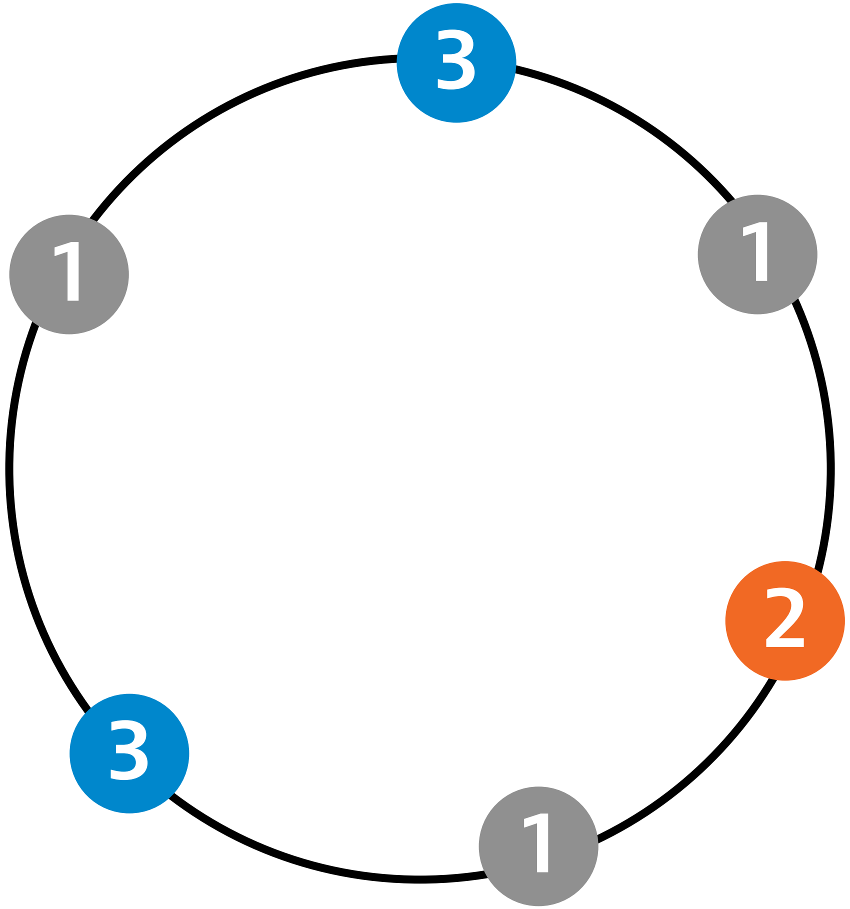

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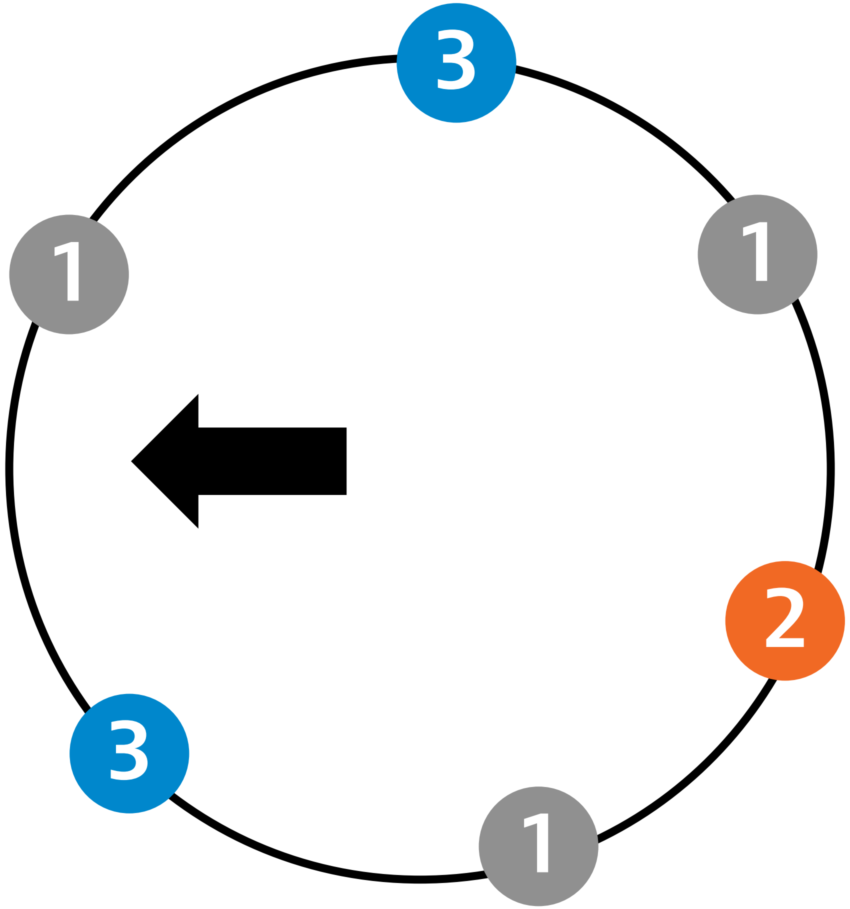

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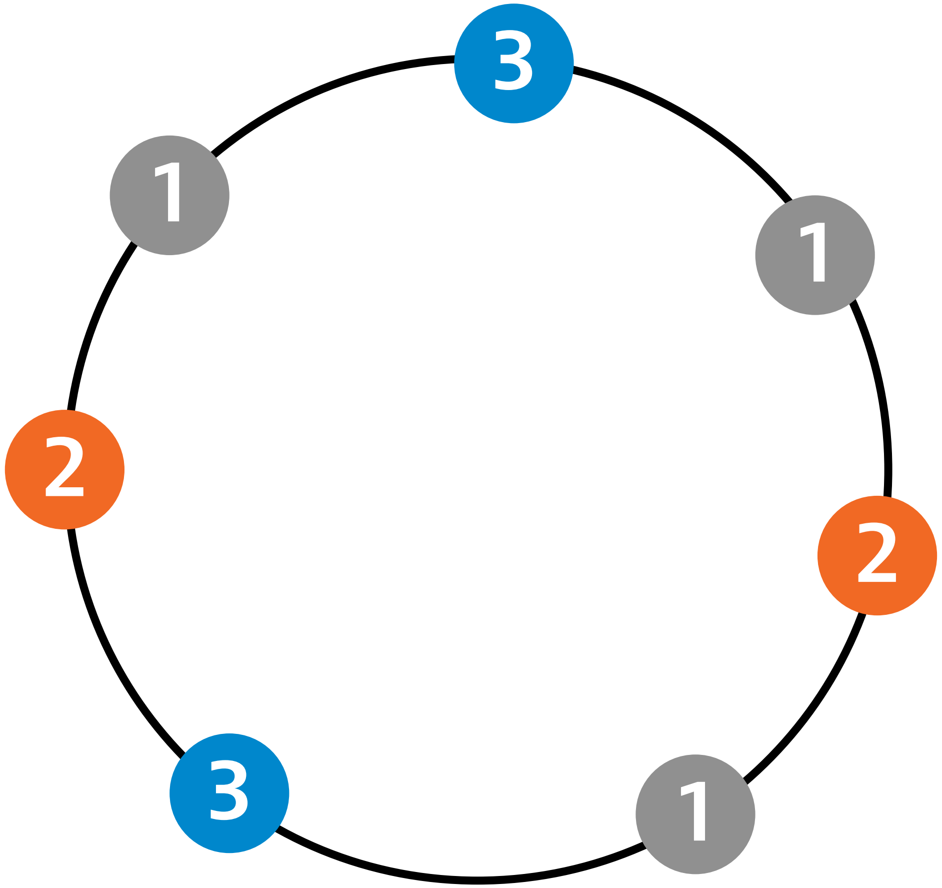

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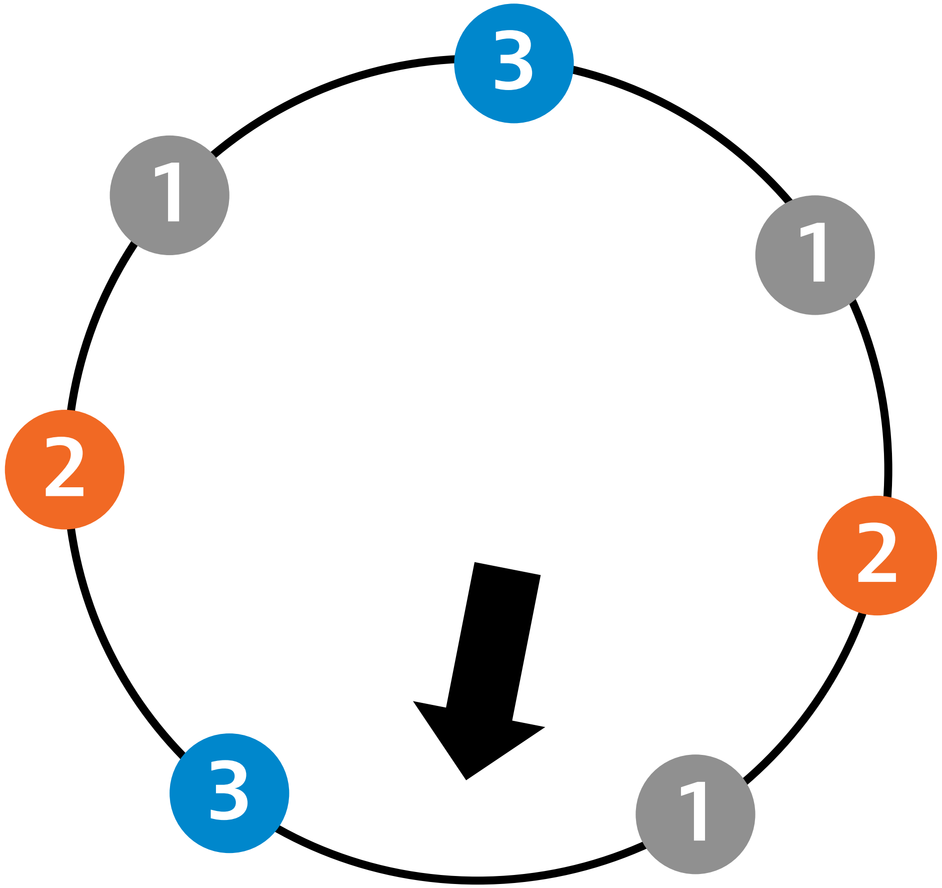

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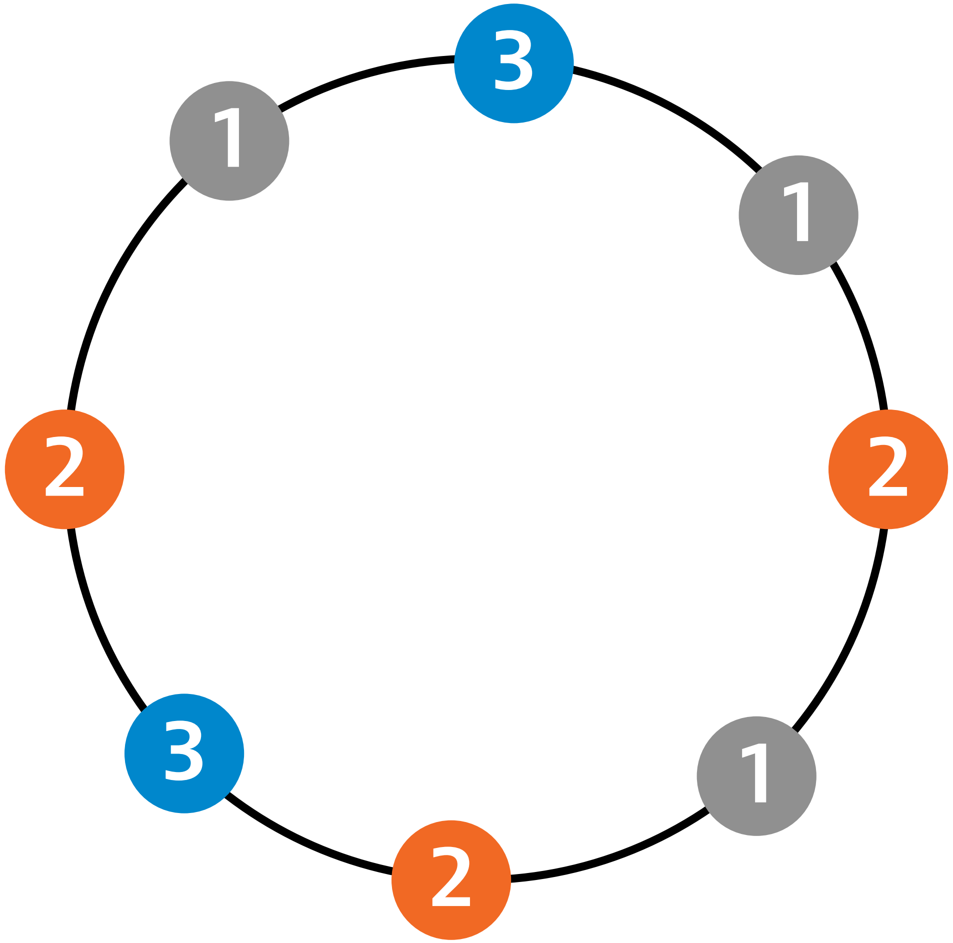

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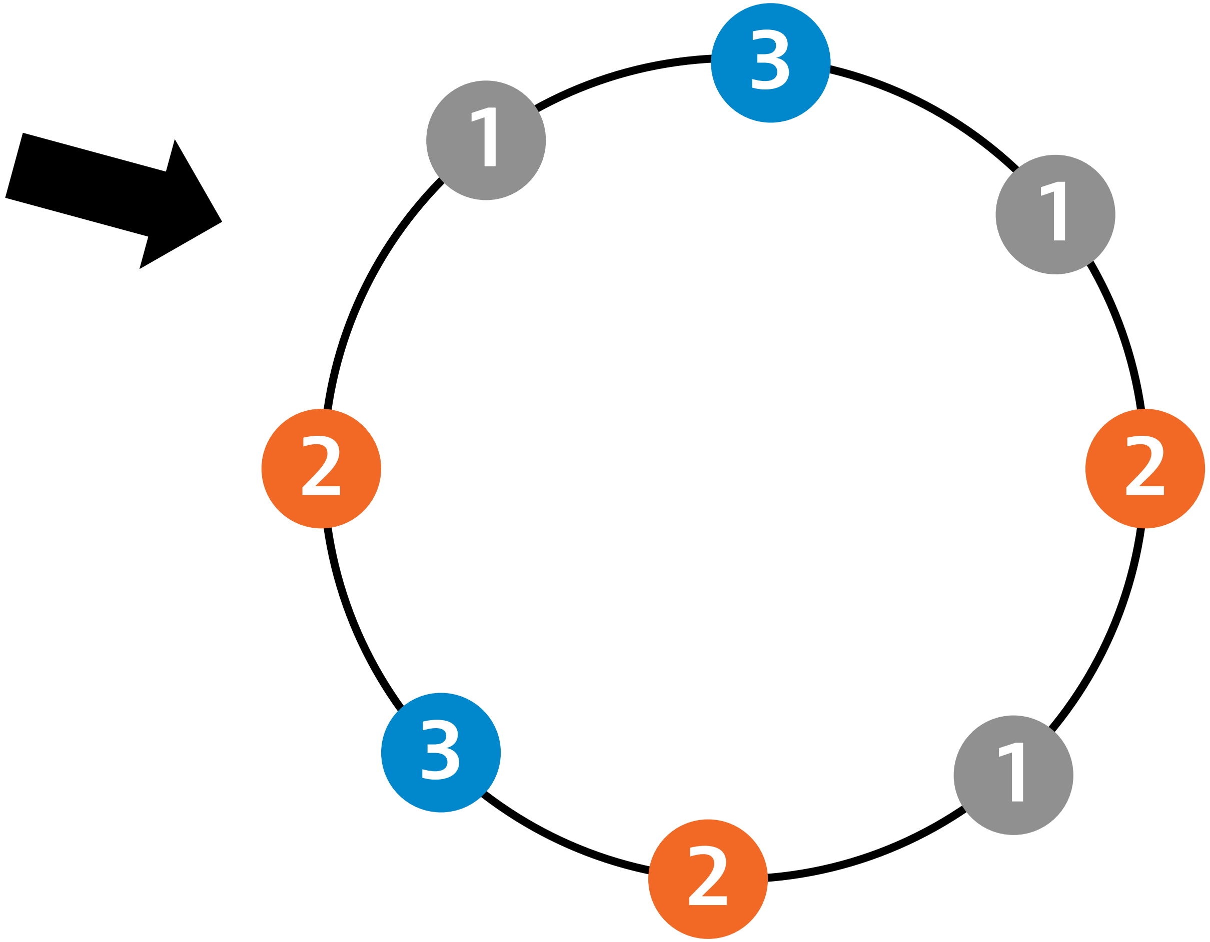

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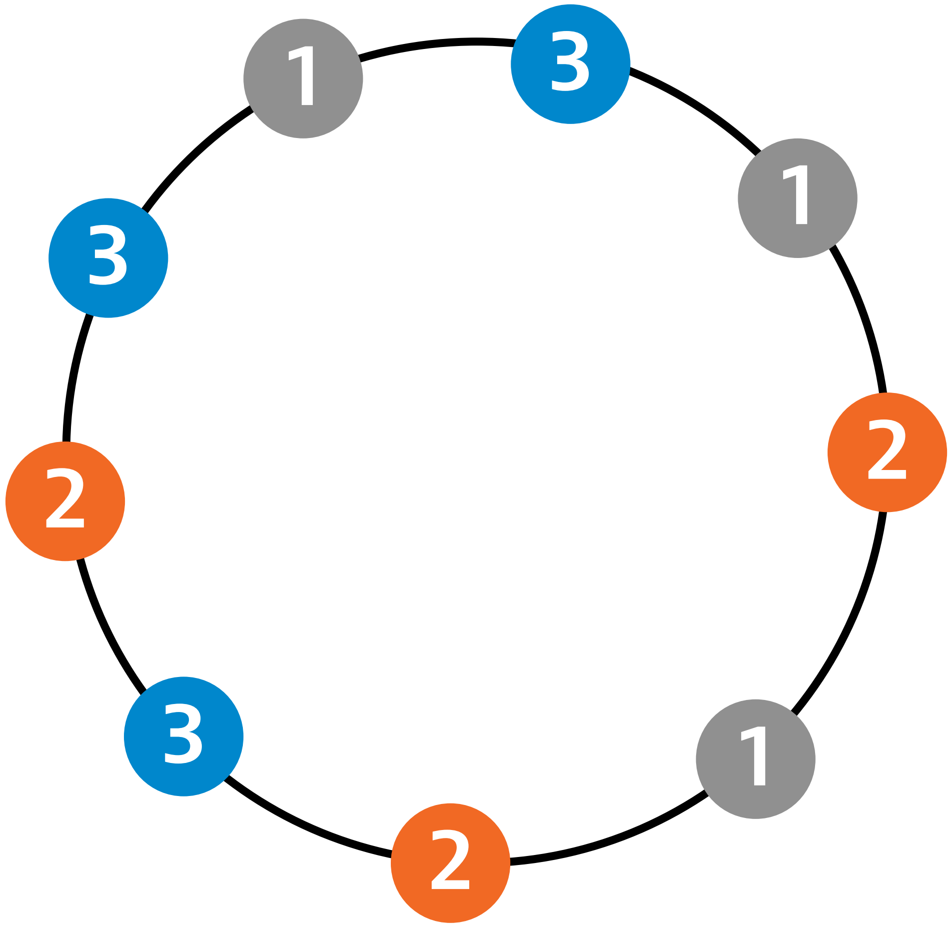

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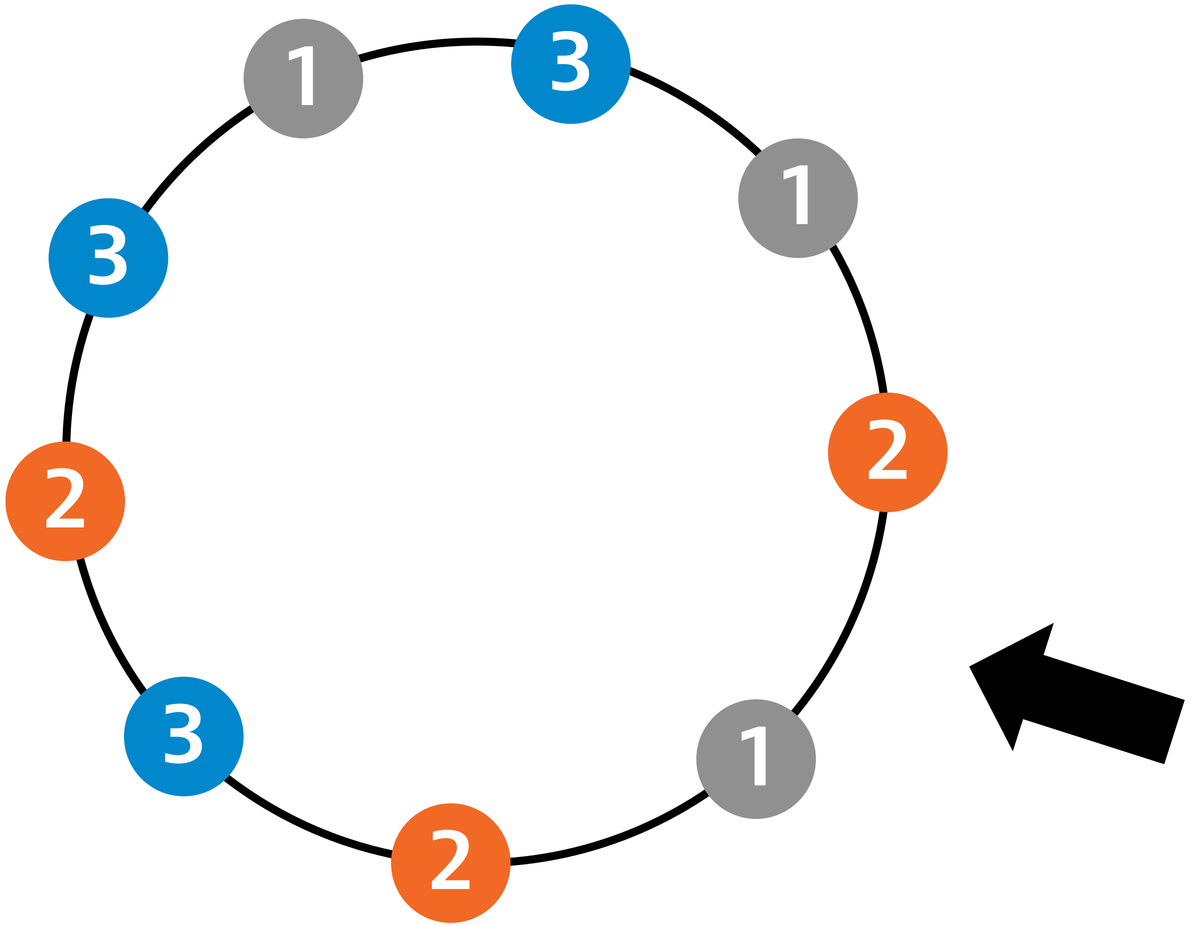

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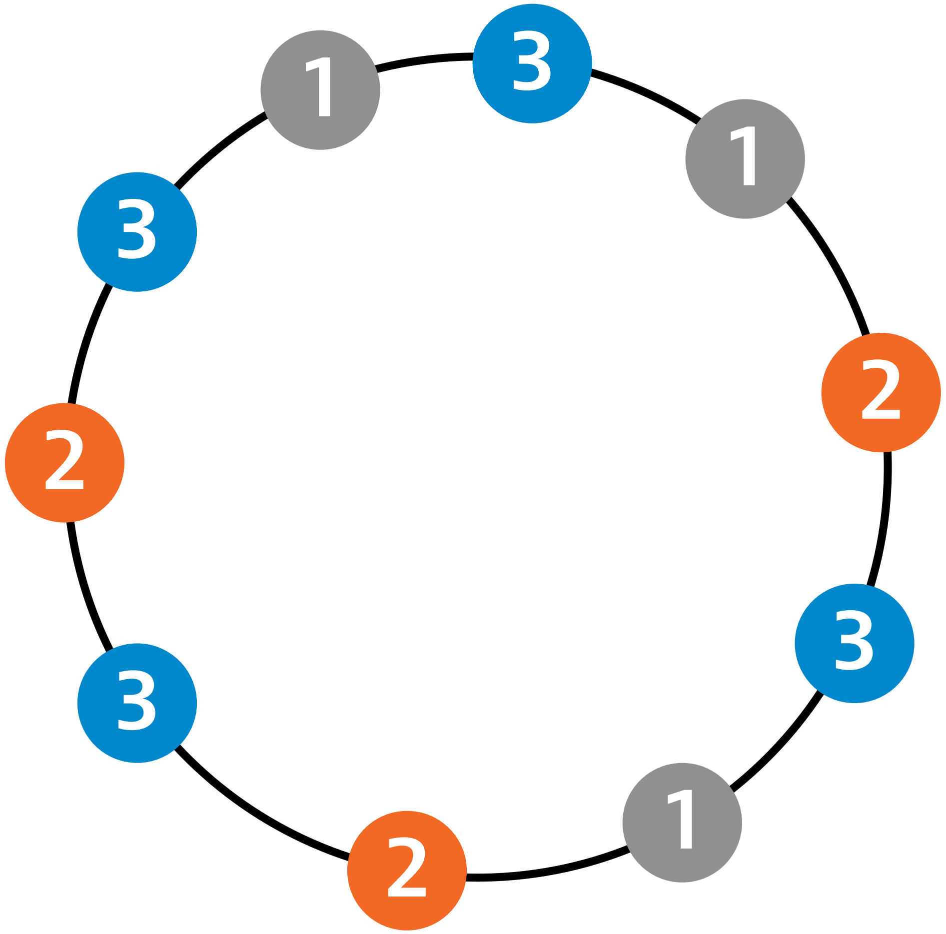

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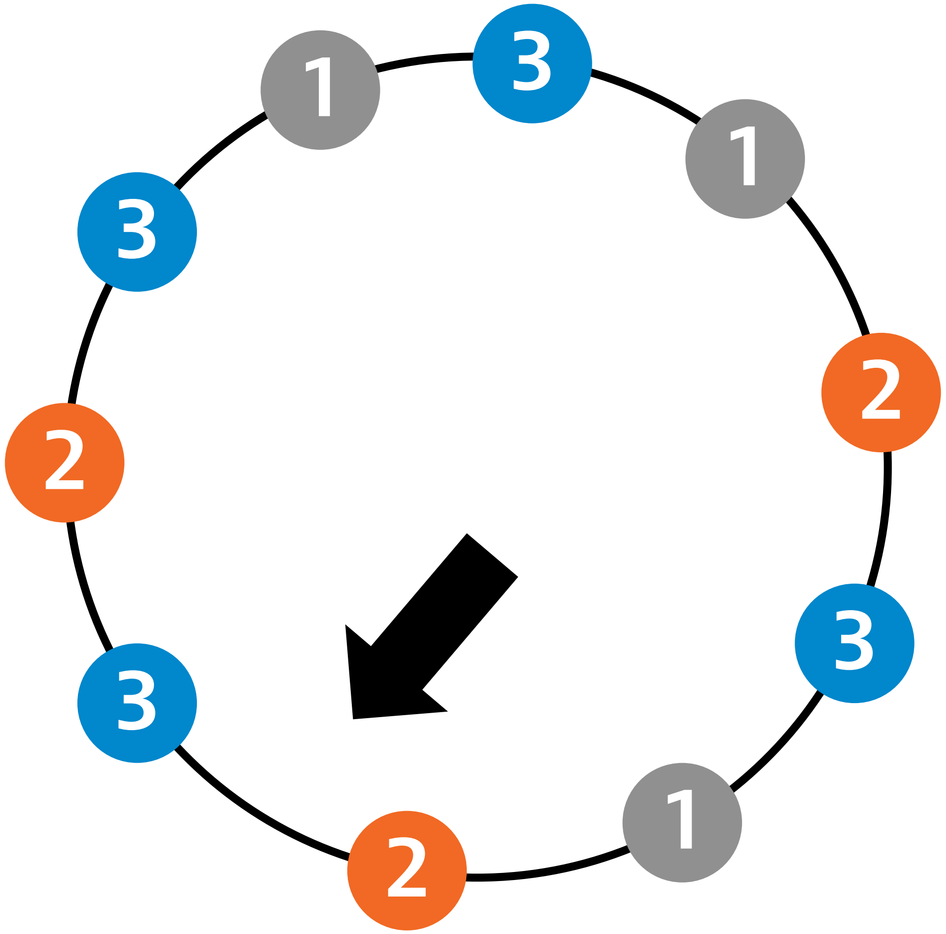

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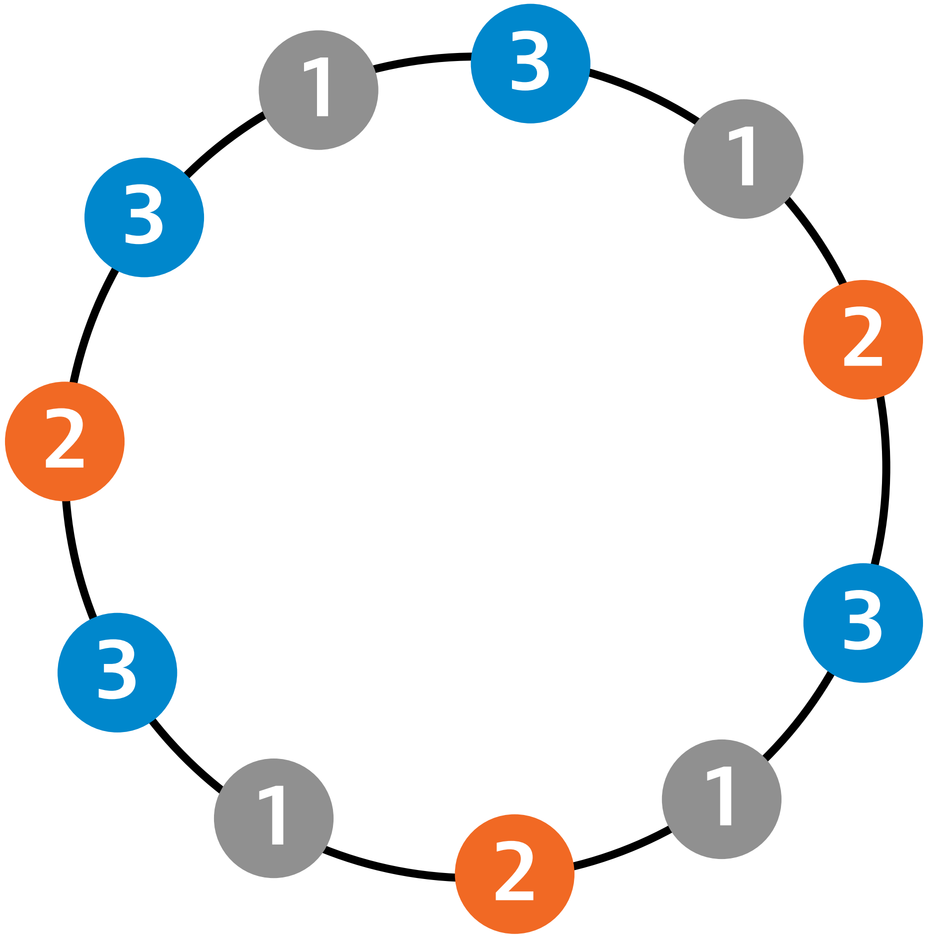

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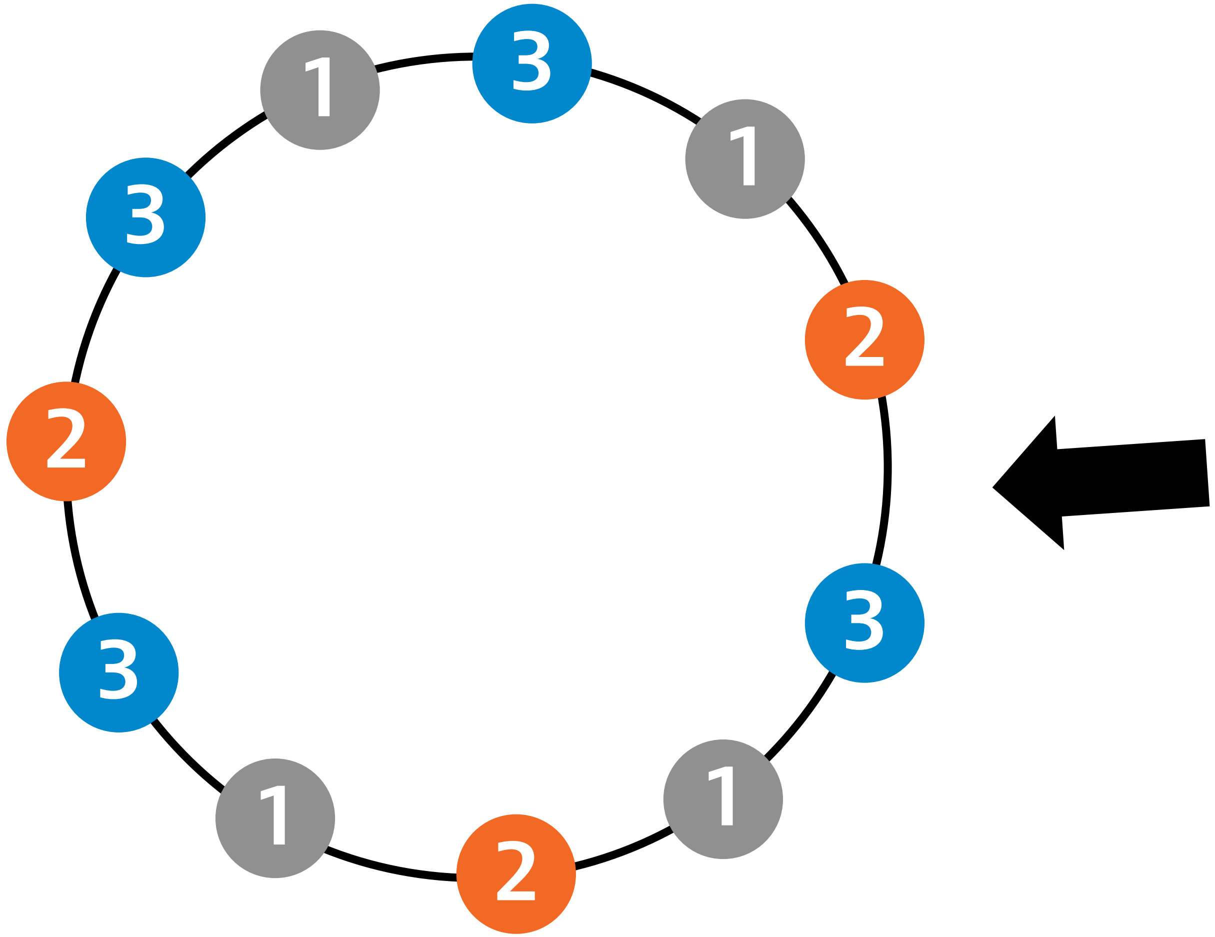

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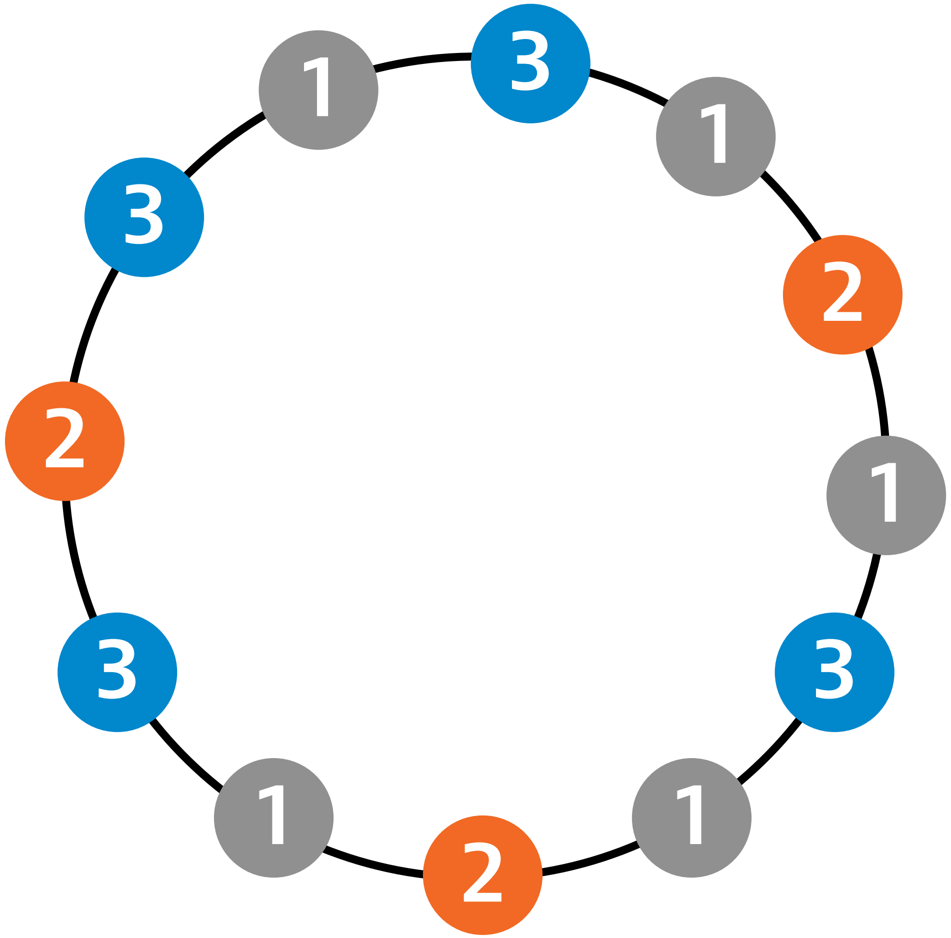

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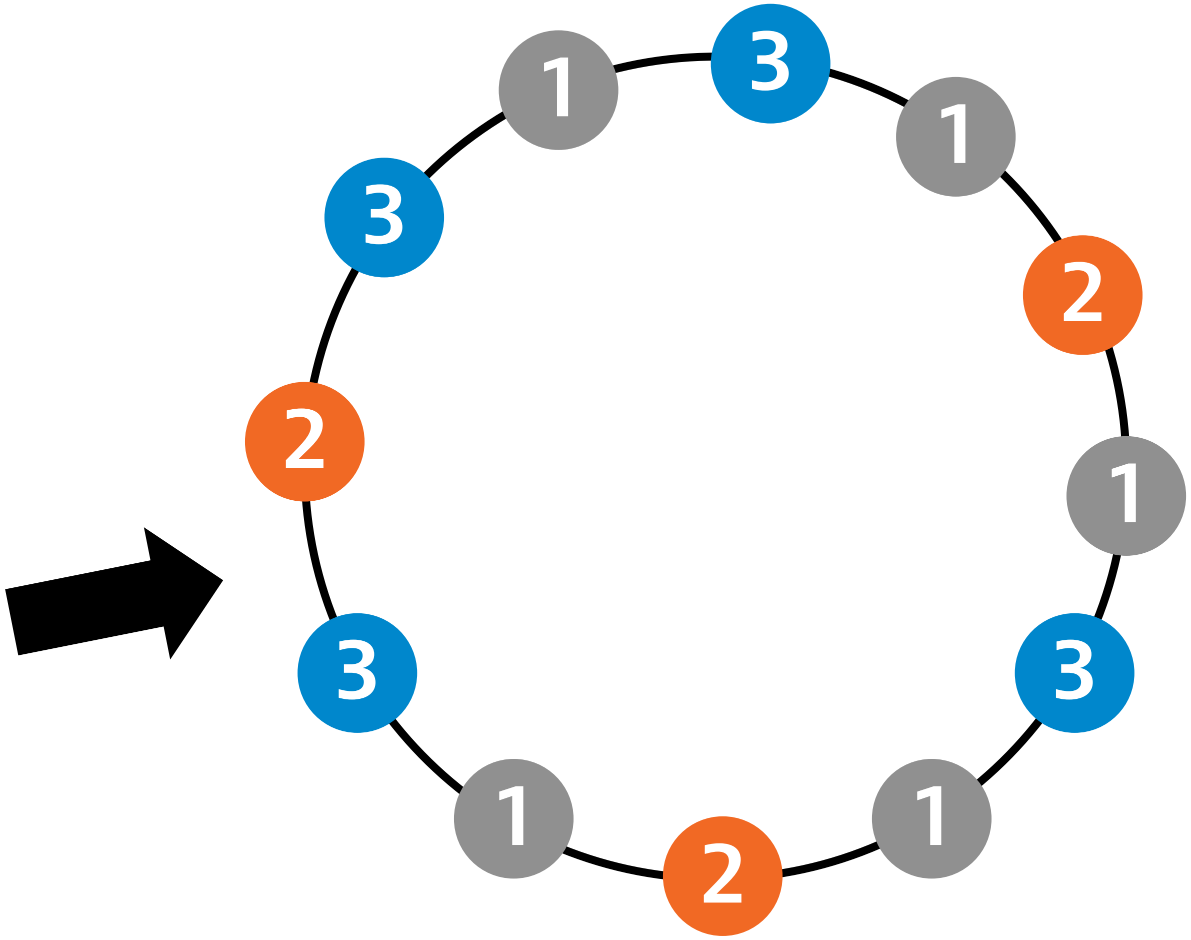

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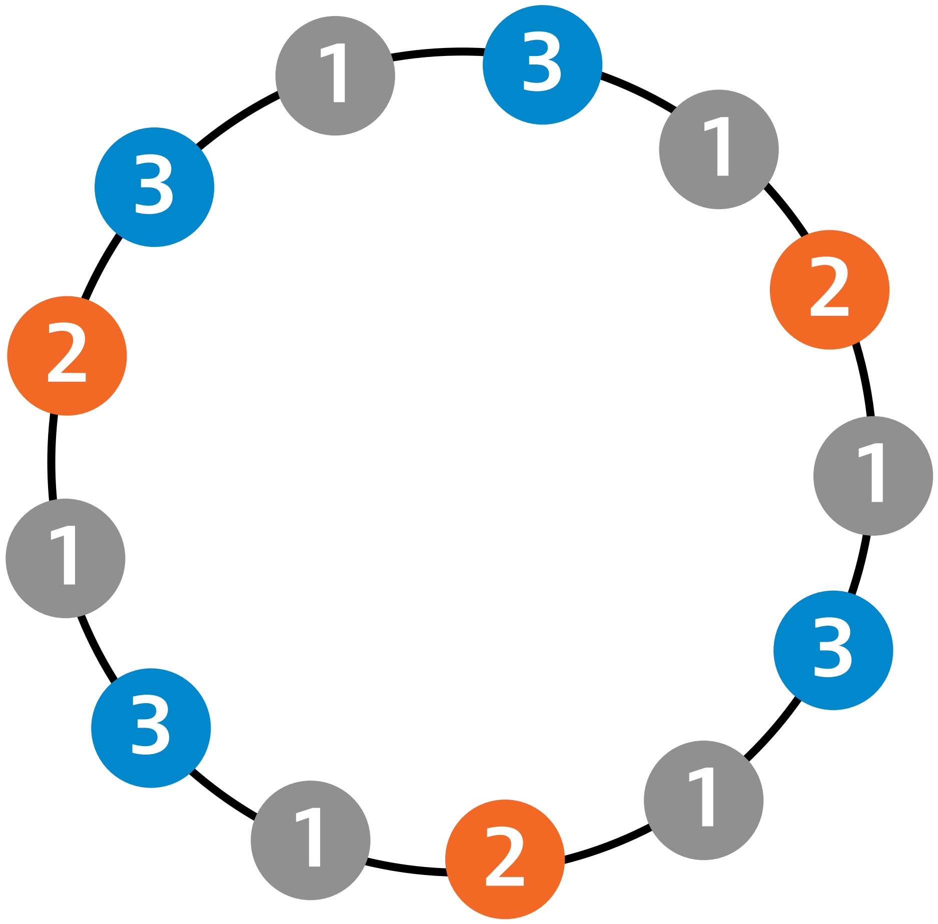

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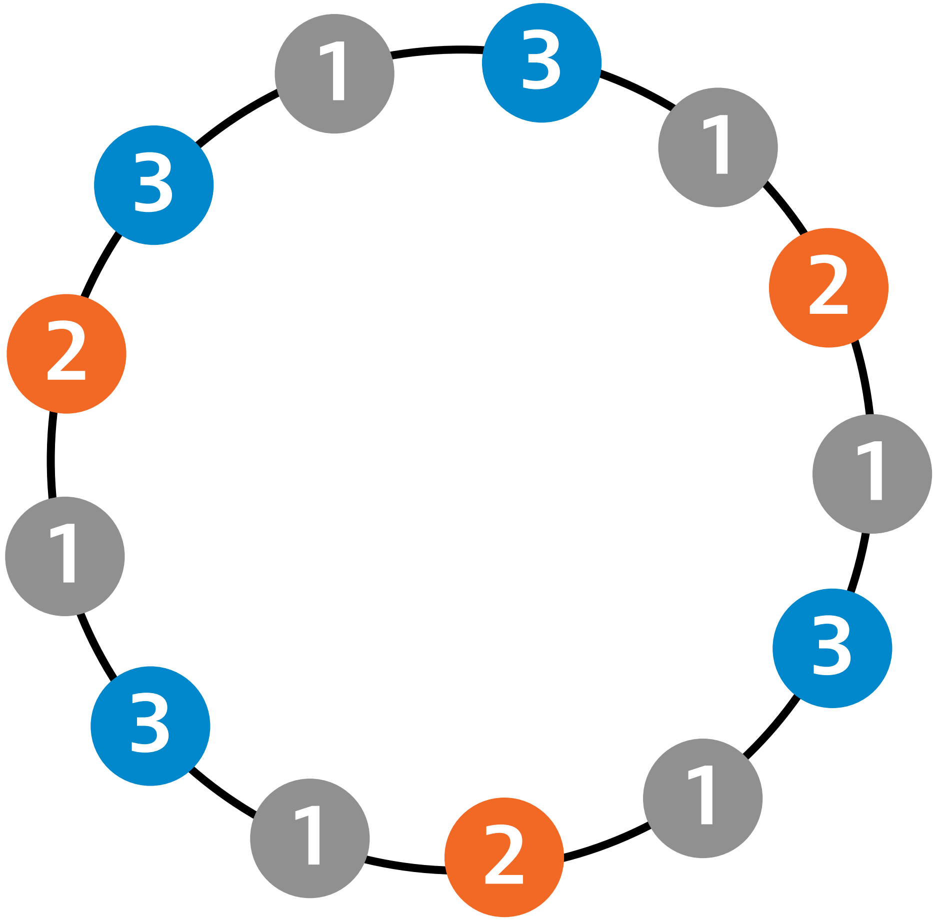
Trivial: stop after n steps → properly 3-colored n-cycle



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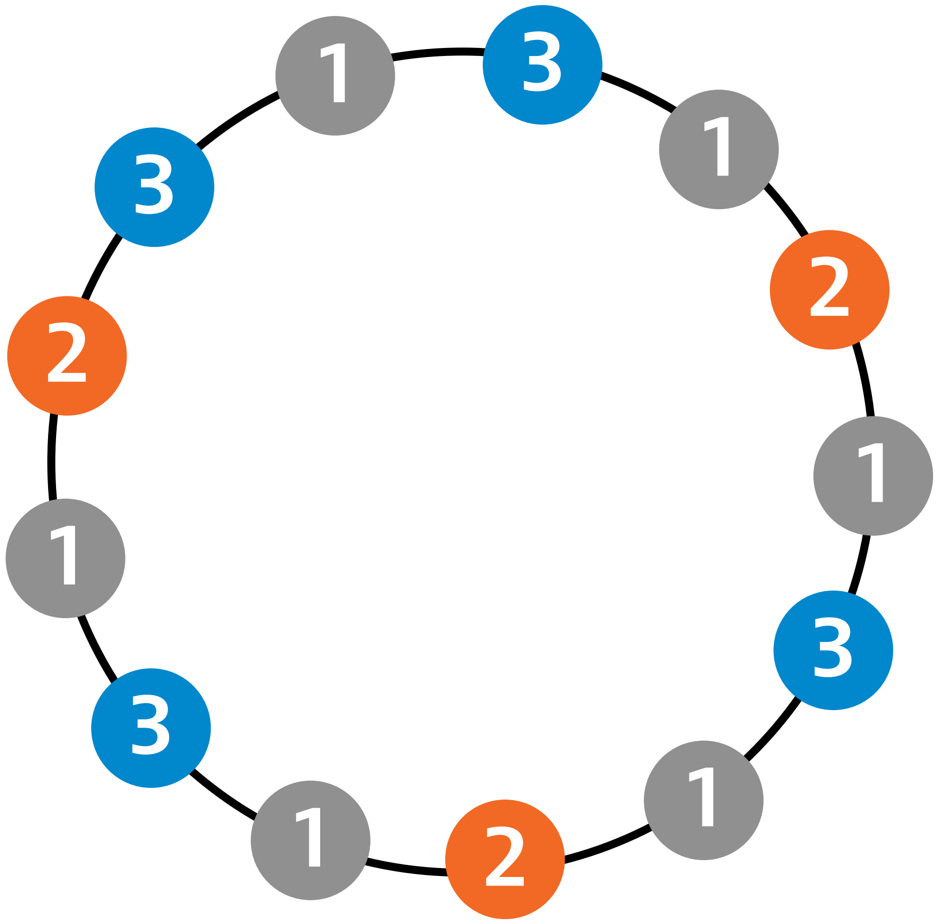

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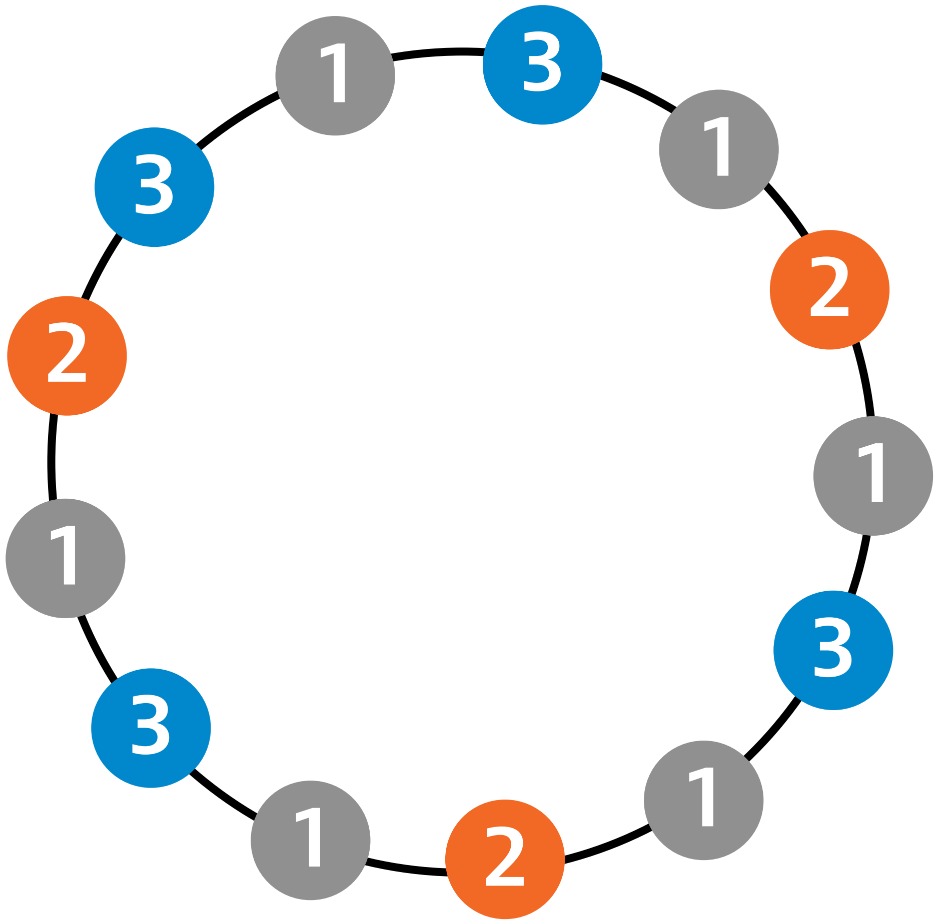

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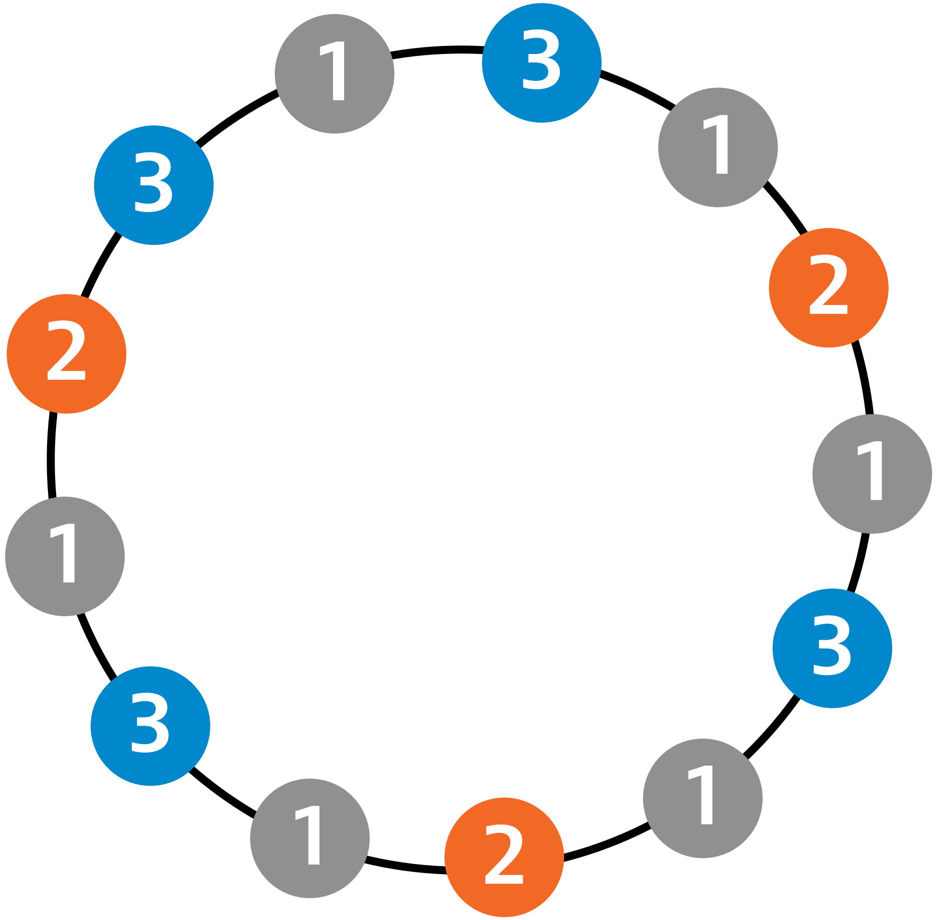

Same for 4 colors: distribution is 1-dependent!

Same for 5 colors: distribution is ***not*** finitely dependent!!!

- **Known:** constant-round *classical or quantum* algorithm → *finitely dependent distribution*
- **Open:** are there problems that admit finitely dependent distributions but no quantum algorithm?
- **Open:** when does a finitely dependent distribution exist?
- **Key example:** coloring cycles